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λ-DEFINABILITY OF BUILT-IN McCARTHY FUNCTIONS AS FUNCTIONS WITH INDETERMINATE VALUES OF ARGUMENTS

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The built-in functions of programming languages are functions with indeterminate values of arguments. The built-in McCarthy functions *car, cdr, cons, null, atom, if, eq, not, and, or*, are used in all functional programming languages. In this paper we show the λ-definability of the built-in McCarthy functions as functions with indeterminate values of arguments. This result is necessary when translating typed functional programming languages into untyped functional programming languages.

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Built-in McCarthy Functions, as Functions with Indeterminate Values of Arguments. Let N=$\{0,1,2,\dots\}$ be the set of natural numbers, each natural number n$\in$N will be called an atom. Let us define the set of symbolic expressions, which we denote by S-expressions [1].

1. n$\in$N $\Rightarrow$ n$\in$S-expressions,
2. $m_1,\dots,m_k$$\in$S-expressions, k$\geq$0 $\Rightarrow$ $(m_1\dots m_k)$$\in$S-expressions and is called a list.

If k=0, than the list ( ) will be called the empty list. We denote the empty list by the atom 0. Thus the empty list will be both an atom and a list.

Let M=S-expressions $\cup\{\perp\}$, where $\perp$ is the element which corresponds to indeterminate value. A mapping $\varphi : M^k \to M$, k$\geq$1, is said to be function with indeterminate values of arguments. Let us define built-in McCarthy functions car, cdr, $cons$, $null$, $atom$, if, eq, not, and, or, as functions with indeterminate values of arguments.

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car, *cdr*, *null*, *atom*, *not* : M→M, and for any m∈M we have:

$$car(\mathrm{m}) = \begin{cases} \mathrm{m}_1, & \text{if } \mathrm{m}=(\mathrm{m}_1\ldots\mathrm{m}_k)\text{, where } \mathrm{m}_1,\ldots,\mathrm{m}_k\in\text{S-expressions, } \mathrm{k}\geq 1, \\ \bot, & \text{otherwise.} \end{cases}$$

$$cdr(\mathrm{m}) = \begin{cases} (\mathrm{m}_2\ldots\mathrm{m}_k), & \text{if } \mathrm{m}=(\mathrm{m}_1\ldots\mathrm{m}_k)\text{, where } \mathrm{m}_1,\ldots,\mathrm{m}_k\in\text{S-expressions, } \mathrm{k}\geq 1, \\ \bot, & \text{otherwise.} \end{cases}$$

$$null(\mathrm{m}) = \begin{cases} 1, & \text{if } \mathrm{m}=0, \\ 0, & \text{if } \mathrm{m}\in\text{S-expressions and } \mathrm{m}\neq 0, \\ \bot, & \text{if } \mathrm{m}=\bot. \end{cases}$$

$$atom(\mathrm{m}) = \begin{cases} 1, & \text{if } \mathrm{m}\in\mathrm{N}, \\ 0, & \text{if } \mathrm{m}\in\text{S-expressions and } \mathrm{m}\notin\mathrm{N}, \\ \bot, & \text{if } \mathrm{m}=\bot. \end{cases}$$

$not(\mathrm{m}) = null(\mathrm{m})$.

cons : $\mathrm{M}^2\to\mathrm{M}$, and for any $\mathrm{m}_0, \mathrm{m}\in\mathrm{M}$ we have:

$$cons(\mathrm{m}_0,\mathrm{m}) = \begin{cases} (\mathrm{m}_0\,\mathrm{m}_1\ldots\mathrm{m}_k), & \text{if } \mathrm{m}_0\in\text{S-expressions, } \mathrm{m}=(\mathrm{m}_1\ldots\mathrm{m}_k), \\ & \text{where } \mathrm{m}_1,\ldots,\mathrm{m}_k\in\text{S-expressions, } \mathrm{k}\geq 0, \\ \bot, & \text{otherwise.} \end{cases}$$

and, *or*, *eq* : $\mathrm{M}^2\to\mathrm{M}$, and for any $\mathrm{m}_1, \mathrm{m}_2\in\mathrm{M}$ we have:

$$and(\mathrm{m}_1,\mathrm{m}_2) = \begin{cases} \mathrm{m}_1, & \text{if } \mathrm{m}_1=0, \\ \mathrm{m}_2, & \text{if } \mathrm{m}_1\in\text{S-expressions and } \mathrm{m}_1\neq 0, \\ \bot, & \text{if } \mathrm{m}_1=\bot. \end{cases}$$

$$or(\mathrm{m}_1,\mathrm{m}_2) = \begin{cases} \mathrm{m}_2, & \text{if } \mathrm{m}_1=0, \\ \mathrm{m}_1, & \text{if } \mathrm{m}_1\in\text{S-expressions and } \mathrm{m}_1\neq 0, \\ \bot, & \text{if } \mathrm{m}_1=\bot. \end{cases}$$

$$eq(\mathrm{m}_1,\mathrm{m}_2) = \begin{cases} 1, & \text{if } \mathrm{m}_1,\mathrm{m}_2\in\mathrm{N} \text{ and } \mathrm{m}_1=\mathrm{m}_2, \\ 0, & \text{if } \mathrm{m}_1,\mathrm{m}_2\in\mathrm{N} \text{ and } \mathrm{m}_1\neq\mathrm{m}_2, \\ \bot, & \text{otherwise.} \end{cases}$$

if : $\mathrm{M}^3\to\mathrm{M}$, and for any $\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3\in\mathrm{M}$ we have:

$$if(\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3) = \begin{cases} \mathrm{m}_2, & \text{if } \mathrm{m}_1\in\text{S-expressions and } \mathrm{m}_1\neq 0, \\ \mathrm{m}_3, & \text{if } \mathrm{m}_1=0, \\ \bot, & \text{if } \mathrm{m}_1=\bot. \end{cases}$$

Untyped λ-Terms, β-Reduction, β-Equality. The definitions of this section can be found in [2]. Let us fix a countable set of variables V. The set Λ of terms is defined as follows:

1. If $x \in V$, then $x \in \Lambda$,
2. If $t_1, t_2 \in \Lambda$, then $(t_1 t_2) \in \Lambda$ (the operation of application),
3. If $x \in V$ and $t \in \Lambda$, then $(\lambda x t) \in \Lambda$ (the operation of abstraction).

We use the following shorthand notations: a term $(\ldots(t_1 t_2)\ldots t_k)$, where $t_i \in \Lambda$, $i = 1,\ldots,k,\ k \geq 2$, is denoted by $t_1 t_2 \ldots t_k$, and a term $(\lambda x_1 (\lambda x_2 (\ldots(\lambda x_n t)\ldots)))$, where $x_j \in V, j = 1,\ldots,n,\ n \geq 1, t \in \Lambda$, is denoted by $\lambda x_1 x_2 \ldots x_n.t$.

The notions of free and bound occurrences of variables in terms as well as the notion of free variable are introduced in the conventional way. The set of all free variables of a term t is denoted by FV(t). A term which does not contain free variables is called a closed term. Terms t_1 and t_2 are said to be congruent (which is denoted by $t_1 \equiv t_2$) if one term can be obtained from the other by renaming bound variables. In what follows, congruent terms are considered identical.

To show a variable x of interest of a term t, the notation t[x] is used. The notation $t[\tau]$ denotes the term obtained by the simultaneous substitution of the term τ for all free occurrences of the variable x. A substitution is said to be admissible if all free variables of the term being substituted remain free after the substitution. We will consider only admissible substitutions.

A term of the form $(\lambda x.t[x])\,\tau$, where $x \in V$, $t,\tau \in \Lambda$ is called a β-redex and the term $t[\tau]$ is called its convolution. A one-step β-reduction ($\rightarrow_\beta$), β-reduction ($\rightarrow\!\!\rightarrow_\beta$) and β-equality ($=_\beta$) are defined in a standard way (see [2]). A term containing no β-redexes is called a normal form. The set of all normal forms is denoted by NF, and the set of all closed normal forms is denoted by NF^0. A term t is said to have a normal form if there exists a term τ, such that $\tau \in NF$ and $t =_\beta \tau$. Of the Church-Rosser theorem [2] it follows:

1. $t =_\beta \tau$ and $\tau \in NF \Rightarrow t \rightarrow\!\!\rightarrow_\beta \tau$,
2. $t =_\beta \tau_1$, $t =_\beta \tau_2$ and $\tau_1, \tau_2 \in NF \Rightarrow \tau_1 \equiv \tau_2$.

If a term has a form $\lambda x_1 \ldots x_k.x t_1 \ldots t_n$, where $x_1,\ldots,x_k, x \in V$, $t_1,\ldots,t_n \in \Lambda$, $k,n \geq 0$, it is called a head normal from and x is called its head variable. The set of all head normal forms is denoted by HNF. A term t is said to have a head normal form if there exists a term τ, such that $\tau \in HNF$ and $t =_\beta \tau$. It is known, that $NF \subset HNF$, but $HNF \not\subset NF$ (see [2]).

P r o p o s i t i o n 1 *[2]. Let $t \in \Lambda$, then: t does not have a head normal form $\Rightarrow$ for any $\tau \in \Lambda$ the term $t\tau$ will not have a head normal form.*

A term t with a fixed occurrence of a subterm τ_1 is denoted by t_{τ_1}, and a term with this occurrence of τ_1 replaced by a term τ_2 is denoted by t_{τ_2}.

P r o p o s i t i o n 2 *[2]. Let $t, \tau_1, \tau_2 \in \Lambda$, then: $\tau_1 =_\beta \tau_2 \Rightarrow t_{\tau_1} =_\beta t_{\tau_2}$.*

Consider the following equation: $f = t[f]$, where $f \in V$, $t[f] \in \Lambda$, $FV(t[f]) \subset \{f\}$. The term τ is a solution of this equation if $\tau =_\beta t[\tau]$.

P r o p o s i t i o n 3 *[2]. The term $Y(\lambda f.t[f])$ is a solution of equation $f=t[f]$, where $Y\equiv\lambda h.(\lambda x.h(xx))(\lambda x.h(xx))$ is Curry fixed point combinator, $h,x\in V$.*

λ-Definability of Built-in McCarthy Functions. We introduce notations for some terms: $I\equiv\lambda x.x$, $T\equiv\lambda xy.x$, $F\equiv\lambda xy.y$, $\Omega\equiv(\lambda x.xx)(\lambda x.xx)$, $[\,]\equiv I$, $[t_1,\dots,t_k]\equiv \lambda x.xt_1[t_2,\dots,t_k]$, where $x,y\in V$, $t_i\in\Lambda$, $i=1,\dots,k$, $k\geq 1$. It is easy to see that: $It\rightarrow_\beta t$, $Tt_1t_2\rightarrow\rightarrow_\beta t_1$, $Ft_1t_2\rightarrow\rightarrow_\beta t_2$, $t,t_1,t_2\in\Lambda$, the term Ω does not have a head normal form.

Let M=S-expressions$\cup\{\perp\}$. To each $m\in M$ we associate the term $m'\in\Lambda$ as follows: $0'\equiv I$, $(n+1)'\equiv\lambda x.xFn'$, where $n\in N$; $(m_1\dots m_k)'\equiv[m_1',\dots,m_k']$, where $m_i\in$S-expressions, $i=1,\dots,k$, $k\geq 0$; $\perp'\equiv\Omega$. It is easy to see, that if $m\in$S-expressions, then $m'\in NF^0$, and if $m_1,m_2\in$S-expressions, $m_1\neq m_2$, then m_1' and m_2' are not congruent closed normal forms.

D e f i n i t i o n . *We say that term $\Phi\in\Lambda$ λ-defines (see [3, 4]) the function $\varphi: M^k\rightarrow M$ $(k\geq 1)$ as a function with indeterminate values of arguments, if for all* $m_1,\dots,m_k\in M$ *we have:*

$\varphi(m_1,\dots,m_k)=m$ *and* $m\neq\perp\ \Rightarrow\ \Phi m_1'\dots m_k'=_\beta m'$,

$\varphi(m_1,\dots,m_k)=\perp\ \Rightarrow\ \Phi m_1'\dots m_k'$ *does not have a head normal form.*

T h e o r e m . *For functions car, cdr, cons, null, atom, if, eq, not, and, or there exist terms Car, Cdr, Cons, Null, Atom, If, Eq, Not, And, Or, respectively, which λ-define them.*

P r o o f . Let us show that the term $Null\equiv\lambda x.x(\lambda xyz.z)T0'1'$ λ-defines the function *null*. To do this, show the following:

a) $Null0'\rightarrow\rightarrow_\beta 1'$.

b) $Null(n+1)'\rightarrow\rightarrow_\beta 0'$, where $n\in N$.

c) $Null[m_1',\dots,m_k']\rightarrow\rightarrow_\beta 0'$, where $m_i\in$S-expressions, $i=1,\dots,k$, $k\geq 1$.

d) $Null\Omega$ does not have a head normal form.

(a) $Null0'\equiv(\lambda x.x(\lambda xyz.z)T0'1')I\rightarrow_\beta I(\lambda xyz.z)T0'1'\rightarrow_\beta(\lambda xyz.z)T0'1'\rightarrow\rightarrow_\beta 1'$.

(b) $Null(n+1)'\equiv(\lambda x.x(\lambda xyz.z)T0'1')(n+1)'\rightarrow_\beta(n+1)'(\lambda xyz.z)T0'1'\equiv$
$(\lambda x.xFn')(\lambda xyz.z)T0'1'\rightarrow_\beta(\lambda xyz.z)Fn'T0'1'\rightarrow\rightarrow_\beta T0'1'\rightarrow\rightarrow_\beta 0'$.

(c) $Null[m_1',\dots,m_k']\equiv(\lambda x.x(\lambda xyz.z)T0'1')[m_1',\dots,m_k']\rightarrow_\beta$
$[m_1',\dots,m_k'](\lambda xyz.z)T0'1'\equiv(\lambda x.xm_1'[m_2',\dots,m_k'])(\lambda xyz.z)T0'1'\rightarrow_\beta$
$(\lambda xyz.z)m_1'[m_2',\dots,m_k']T0'1'\rightarrow\rightarrow_\beta T0'1'\rightarrow\rightarrow_\beta 0'$.

(d) $Null\Omega\equiv(\lambda x.x(\lambda xyz.z)T0'1')\Omega\rightarrow_\beta\Omega(\lambda xyz.z)T0'1'$ and, according to Proposition 1, the term $Null\Omega$ does not have a head normal form.

Let us show that the term $If\equiv\lambda xyz.(Nullx)Tyz$ λ-defines the function *if*. To do this, show the following:

a) $If0'm_2'm_3'\rightarrow\rightarrow_\beta m_3'$, where $m_2,m_3\in M$.

b) $If(n+1)'m_2'm_3'\rightarrow\rightarrow_\beta m_2'$, where $n\in N$, $m_2,m_3\in M$.

c) $If[\mu_1',\ldots,\mu_k']m_2'm_3'\to\to_\beta m_2'$, where $\mu_i\in$S-expressions, i=1, …, k, k≥1, $m_2, m_3\in M$.

d) $If\Omega m_2'm_3'$, where $m_2, m_3\in M$, does not have a head normal form.

(a) $If0'm_2'm_3'\equiv(\lambda xyz.(Null\,x)Tyz)0'm_2'm_3'\to\to_\beta(Null0')Tm_2'm_3'\to\to_\beta$ $1'Tm_2'm_3'\equiv(\lambda x.xF0')Tm_2'm_3'\to_\beta(TF0')m_2'm_3'\to\to_\beta Fm_2'm_3'\to\to_\beta m_3'$.

(b) $If(n+1)'m_2'm_3'\equiv(\lambda xyz.(Null\,x)Tyz)(n+1)'m_2'm_3'\to\to_\beta$ $(Null(n+1)')Tm_2'm_3'\to\to_\beta 0'Tm_2'm_3'\to_\beta Tm_2'm_3'\to\to_\beta m_2'$.

(c) $If[\mu_1',\ldots,\mu_k']m_2'm_3'\equiv(\lambda xyz.(Null\,x)Tyz)[\mu_1',\ldots,\mu_k']m_2'm_3'\to\to_\beta$ $(Null[\mu_1',\ldots,\mu_k'])Tm_2'm_3'\to\to_\beta 0'Tm_2'm_3'\to_\beta Tm_2'm_3'\to\to_\beta m_2'$.

(d) $If\Omega m_2'm_3'\equiv(\lambda xyz.(Null\,x)Tyz)\Omega m_2'm_3'\to\to_\beta(Null\Omega)Tm_2'm_3'$ and, according to Proposition 1, the term $If\Omega m_2'm_3'$ does not have a head normal form.

Let us show that the term $Atom\equiv\lambda x.If(Null\,x)1'(False(xT))$, where $False\equiv\lambda x.x(\lambda xyz.0')T1'F$, λ-defines the function *atom*. To do this, show the following:

a) $Atom0'\to\to_\beta 1'$.

b) $Atom(n+1)'\to\to_\beta 1'$, where n∈N.

c) $Atom[m_1',\ldots,m_k']\to\to_\beta 0'$, where $m_i\in$S-expressions, i=1, …, k, k≥1.

d) $Atom\Omega$ does not have a head normal form.

(a) $Atom0'\equiv(\lambda x.If(Null\,x)1'(False(xT)))0'\to_\beta If(Null0')1'(False(0'T))\to\to_\beta$ $If1'1'(False(0'T))\to\to_\beta 1'$

(b) $Atom(n+1)'\equiv(\lambda x.If(Null\,x)1'(False(xT)))(n+1)'\to_\beta$ $If(Null(n+1)')1'(False((n+1)'T))\to\to_\beta If0'1'(False((n+1)'T))\to\to_\beta$ $False((n+1)'T)\equiv False((\lambda x.xFn')T)\to_\beta False(TFn')\to\to_\beta False\,F\equiv$ $(\lambda x.x(\lambda xyz.0')T1'F)F\to_\beta F(\lambda xyz.0')T1'F\to\to_\beta T1'F\to\to_\beta 1'$.

(c) $Atom[m_1',\ldots,m_k']\equiv(\lambda x.If(Null\,x)1'(False(xT)))[m_1',\ldots,m_k']\to_\beta$ $If(Null[m_1',\ldots,m_k'])1'(False([m_1',\ldots,m_k']T))\to\to_\beta$ $If0'1'(False([m_1',\ldots,m_k']T))\to\to_\beta False([m_1',\ldots,m_k']T)\equiv$ $False((\lambda x.xm_1'[m_2',\ldots,m_k'])T)\to_\beta False(Tm_1'[m_2',\ldots,m_k'])\to\to_\beta False\,m_1'$.

There are three possible cases: c1) $m_1'\equiv 0'$, c2) $m_1'\equiv(n+1)'$, where n∈N, c3) $m_1'\equiv[\mu_1',\ldots,\mu_s']$, where $\mu_i\in$S-expressions, i=1, …, s, s≥1.

(c1) $False0'\equiv(\lambda x.x(\lambda xyz.0')T1'F)I\to_\beta I(\lambda xyz.0')T1'F\to_\beta$ $(\lambda xyz.0')T1'F\to\to_\beta 0'$.

(c2) $False(n+1)'\equiv(\lambda x.x(\lambda xyz.0')T1'F)(n+1)'\to_\beta(n+1)'(\lambda xyz.0')T1'F\equiv$ $(\lambda x.xFn')(\lambda xyz.0')T1'F\to_\beta(\lambda xyz.0')Fn'T1'F\to\to_\beta 0'1'F\to_\beta 1'F\equiv$ $(\lambda x.xF0')F\to_\beta FF0'\to\to_\beta 0'$.

(c3) $False[\mu_1',\ldots,\mu_s']\equiv(\lambda x.x(\lambda xyz.0')T1'F)[\mu_1',\ldots,\mu_s']\to_\beta$ $[\mu_1',\ldots,\mu_s'](\lambda xyz.0')T1'F\equiv(\lambda x.x\mu_1'[\mu_2',\ldots,\mu_s'])(\lambda xyz.0')T1'F\to_\beta$ $(\lambda xyz.0')\mu_1'[\mu_2',\ldots,\mu_s']T1'F\to\to_\beta 0'1'F\to_\beta 1'F\equiv$

$(\lambda x.xF0')F \to_\beta FF0' \to\to_\beta 0'$.

(d) $Atom\Omega \equiv (\lambda x.If(Null\,x)1'(False(xT)))\Omega \to_\beta If(Null\Omega)1'(False(\Omega T)) \equiv$ $(\lambda xyz.(Null\,x)Tyz)(Null\Omega)1'(False(\Omega T)) \to\to_\beta Null(Null\Omega)T1'(False(\Omega T)) \equiv$ $(\lambda x.x(\lambda xyz.z)T0'1')(Null\Omega)T1'(False(\Omega T)) \to_\beta$ $(Null\Omega)(\lambda xyz.z)T0'1'T1'(False(\Omega T))$ and, according to Proposition 1, the term $Atom\Omega$ does not have a head normal form.

Since *not*=*null*, then $Not \equiv Null$.

Let us show that the term $And \equiv \lambda xy.If(Null\,x)xy$ λ-defines the function *and*. To do this, show the following:

a) $And0'm_2' \to\to_\beta 0'$, where $m_2 \in M$.

b) $And(n+1)'m_2' \to\to_\beta m_2'$, where $n \in N$, $m_2 \in M$.

c) $And[\mu_1', \ldots, \mu_k']m_2' \to\to_\beta m_2'$, where $\mu_i \in$S-expressions, $i=1, \ldots, k$, $k \geq 1$, $m_2 \in M$.

d) $And\Omega m_2'$, where $m_2 \in M$, does not have a head normal form.

(a) $And0'm_2' \equiv (\lambda xy.If(Null\,x)xy)0'm_2' \to\to_\beta If(Null0')0'm_2' \to\to_\beta If1'0'm_2' \to\to_\beta 0'$.

(b) $And(n+1)'m_2' \equiv (\lambda xy.If(Null\,x)xy)(n+1)'m_2' \to\to_\beta If(Null(n+1)')(n+1)'m_2' \to\to_\beta$ $If0'(n+1)'m_2' \to\to_\beta m_2'$.

(c) $And[\mu_1', \ldots, \mu_k']m_2' \equiv (\lambda xy.If(Null\,x)xy)[\mu_1', \ldots, \mu_k']m_2' \to\to_\beta$ $If(Null[\mu_1', \ldots, \mu_k'])[\mu_1', \ldots, \mu_k']m_2' \to\to_\beta If0'[\mu_1', \ldots, \mu_k']m_2' \to\to_\beta m_2'$.

(d) $And\Omega m_2' \equiv (\lambda xy.If(Null\,x)xy)\Omega m_2' \to\to_\beta If(Null\Omega)\Omega m_2' \equiv$ $(\lambda xyz.(Null\,x)Tyz)(Null\Omega)\Omega m_2' \to\to_\beta Null(Null\Omega)T\Omega m_2' \equiv$ $(\lambda x.x(\lambda xyz.z)T0'1')(Null\Omega)T\Omega m_2' \to_\beta (Null\Omega)(\lambda xyz.z)T0'1'T\Omega m_2'$ and, according to Proposition 1, the term $And\Omega m_2'$ does not have a head normal form.

Let us show that the term $Or \equiv \lambda xy.If(Null\,x)yx$ λ-defines the function *or*. To do this, show the following:

a) $Or0'm_2' \to\to_\beta m_2'$, where $m_2 \in M$.

b) $Or(n+1)'m_2' \to\to_\beta (n+1)'$, where $n \in N$, $m_2 \in M$.

c) $Or[\mu_1', \ldots, \mu_k']m_2' \to\to_\beta [\mu_1', \ldots, \mu_k']$, where $\mu_i \in$S-expressions, $i=1, \ldots, k$, $k \geq 1$, $m_2 \in M$.

d) $Or\Omega m_2'$, where $m_2 \in M$, does not have a head normal form.

(a) $Or0'm_2' \equiv (\lambda xy.If(Null\,x)yx)0'm_2' \to\to_\beta If(Null0')m_2'0' \to\to_\beta If1'm_2'0' \to\to_\beta m_2'$.

(b) $Or(n+1)'m_2' \equiv (\lambda xy.If(Null\,x)yx)(n+1)'m_2' \to\to_\beta If(Null(n+1)')m_2'(n+1)' \to\to_\beta$ $If0'm_2'(n+1)' \to\to_\beta (n+1)'$.

(c) $Or[\mu_1', \ldots, \mu_k']m_2' \equiv (\lambda xy.If(Null\,x)yx)[\mu_1', \ldots, \mu_k']m_2' \to\to_\beta$ $If(Null[\mu_1', \ldots, \mu_k'])m_2'[\mu_1', \ldots, \mu_k'] \to\to_\beta If0'm_2'[\mu_1', \ldots, \mu_k'] \to\to_\beta$ $[\mu_1', \ldots, \mu_k']$.

(d) $\mathit{Or}\Omega m_2' \equiv (\lambda xy.\mathit{If}(\mathit{Null}x)yx)\Omega m_2' \to\to_\beta \mathit{If}(\mathit{Null}\Omega)m_2'\Omega \equiv$ $(\lambda xyz.(\mathit{Null}x)Tyz)(\mathit{Null}\Omega)m_2'\Omega \to\to_\beta \mathit{Null}(\mathit{Null}\Omega)Tm_2'\Omega \equiv$ $(\lambda x.x(\lambda xyz.z)T0'1')(\mathit{Null}\Omega)Tm_2'\Omega \to_\beta (\mathit{Null}\Omega)(\lambda xyz.z)T0'1'Tm_2'\Omega$ and, according to Proposition 1, the term $\mathit{Or}\Omega m_2'$ does not have a head normal form.

Let us show that the term $\mathit{Car} \equiv \lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xT)$ λ-defines the function *car*. To do this, show the following:

a) $\mathit{Car}[m_1', \dots, m_k'] \to\to_\beta m_1'$, where $m_i \in$S-expressions, $i=1,\dots,k$, $k \geq 1$.

b) $\mathit{Car}n'$, where $n \in N$, does not have a head normal form.

c) $\mathit{Car}\Omega$ does not have a head normal form.

(a) $\mathit{Car}[m_1', \dots, m_k'] \equiv (\lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xT))[m_1', \dots, m_k'] \to_\beta$ $\mathit{If}(\mathit{Atom}[m_1', \dots, m_k'])\Omega([m_1', \dots, m_k']T) \to\to_\beta \mathit{If}0'\Omega([m_1', \dots, m_k']T) \to\to_\beta$ $[m_1', \dots, m_k']T \equiv (\lambda x.xm_1'[m_2', \dots, m_k'])T \to_\beta Tm_1'[m_2', \dots, m_k'] \to\to_\beta m_1'$.

(b) $\mathit{Car}n' \equiv (\lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xT))n' \to_\beta \mathit{If}(\mathit{Atom}n')\Omega(n'T) \to\to_\beta \mathit{If}1'\Omega(n'T) \to\to_\beta \Omega$ and the term $\mathit{Car}n'$ does not have a head normal form.

(c) $\mathit{Car}\Omega \equiv (\lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xT))\Omega \to_\beta \mathit{If}(\mathit{Atom}\Omega)\Omega(\Omega T) \equiv$ $(\lambda xyz.(\mathit{Null}x)Tyz)(\mathit{Atom}\Omega)\Omega(\Omega T) \to\to_\beta \mathit{Null}(\mathit{Atom}\Omega)T\Omega(\Omega T) \equiv$ $(\lambda x.x(\lambda xyz.z)T0'1')(\mathit{Atom}\Omega)T\Omega(\Omega T) \to_\beta (\mathit{Atom}\Omega)(\lambda xyz.z)T0'1'T\Omega(\Omega T)$ and, according to Proposition 1, the term $\mathit{Car}\Omega$ does not have a head normal form.

Let us show that the term $\mathit{Cdr} \equiv \lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xF)$ λ-defines the function *cdr*. To do this, show the following:

a) $\mathit{Cdr}[m_1', \dots, m_k'] \to\to_\beta [m_2', \dots, m_k']$, where $m_i \in$S-expressions, $i=1,\dots,k$, $k \geq 1$.

b) $\mathit{Cdr}n'$, where $n \in N$, does not have a head normal form.

c) $\mathit{Cdr}\Omega$ does not have a head normal form.

(a) $\mathit{Cdr}[m_1', \dots, m_k'] \equiv (\lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xF))[m_1', \dots, m_k'] \to_\beta$ $\mathit{If}(\mathit{Atom}[m_1', \dots, m_k'])\Omega([m_1', \dots, m_k']F) \to\to_\beta \mathit{If}0'\Omega([m_1', \dots, m_k']F) \to\to_\beta$ $[m_1', \dots, m_k']F \equiv (\lambda x.xm_1'[m_2', \dots, m_k'])F \to_\beta$ $Fm_1'[m_2', \dots, m_k'] \to\to_\beta [m_2', \dots, m_k']$.

(b) $\mathit{Cdr}n' \equiv (\lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xF))n' \to_\beta \mathit{If}(\mathit{Atom}n')\Omega(n'F) \to\to_\beta \mathit{If}1'\Omega(n'T) \to\to_\beta \Omega$ and the term $\mathit{Cdr}n'$ does not have a head normal form.

(c) $\mathit{Cdr}\Omega \equiv (\lambda x.\mathit{If}(\mathit{Atom}x)\Omega(xF))\Omega \to_\beta \mathit{If}(\mathit{Atom}\Omega)\Omega(\Omega F) \equiv$ $(\lambda xyz.(\mathit{Null}x)Tyz)(\mathit{Atom}\Omega)\Omega(\Omega F) \to\to_\beta \mathit{Null}(\mathit{Atom}\Omega)T\Omega(\Omega F) \equiv$ $(\lambda x.x(\lambda xyz.z)T0'1')(\mathit{Atom}\Omega)T\Omega(\Omega F) \to_\beta (\mathit{Atom}\Omega)(\lambda xyz.z)T0'1'T\Omega(\Omega F)$ and, according to Proposition 1, the term $\mathit{Cdr}\Omega$ does not have a head normal form.

Let us show that the term

$$\mathit{Cons} \equiv \lambda xy.\mathit{If}(\mathit{And}(\mathit{Atom}y)(\mathit{Not}(\mathit{Null}y)))\Omega(\mathit{If}x(\lambda z.zxy)(\lambda z.zxy))$$

λ-defines the function *cons*. To do this, show the following:

a) $Cons\mathrm{m}_0{}'[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}'] \to\to_\beta [\mathrm{m}_0{}',\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']$, where $\mathrm{m}_i \in$S-expressions, $\mathrm{i}=0,\ldots,\mathrm{k},\ \mathrm{k}\geq 0$.

b) $Cons\mathrm{m}'(\mathrm{n}+1)'$, where $\mathrm{m}\in \mathrm{M}$, $\mathrm{n}\in \mathrm{N}$, does not have a head normal form.

c) $Cons\mathrm{m}'\Omega$, where $\mathrm{m}\in \mathrm{M}$, does not have a head normal form.

d) $Cons\Omega[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']$, where $\mathrm{m}_i\in$S-expressions, $\mathrm{i}=1,\ldots,\mathrm{k},\ \mathrm{k}\geq 0$, does not have a head normal form.

(a) There are two possible cases: a1) k=0, a2) k>0.

(a1) $Cons\mathrm{m}_0{}'[\ ]\equiv$
$(\lambda \mathrm{xy}.If(And(Atom\mathrm{y})(Not(Null\mathrm{y})))\Omega(If\mathrm{x}(\lambda \mathrm{z.zxy})(\lambda \mathrm{z.zxy})))\mathrm{m}_0'[\ ]\to\to_\beta$
$If(And(Atom[\ ])(Not(Null[\ ])))\Omega(If\mathrm{m}_0{}'(\lambda \mathrm{z.zm}_0{}'[\ ])(\lambda \mathrm{z.zm}_0{}'[\ ]))\to\to_\beta$
$If(And1'0')\Omega[\mathrm{m}_0{}'])\to\to_\beta If0'\Omega[\mathrm{m}_0{}']\to\to_\beta[\mathrm{m}_0{}']$.

(a2) $Cons\mathrm{m}_0{}'[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']\equiv(\lambda \mathrm{xy}.If(And(Atom\mathrm{y})(Not(Null\mathrm{y})))\Omega$
$(If\mathrm{x}(\lambda \mathrm{z.zxy})(\lambda \mathrm{z.zxy})))\mathrm{m}_0{}'[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']\to\to_\beta$
$If(And(Atom[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}'])(Not(Null[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}'])))\Omega$
$(If\mathrm{m}_0{}'(\lambda \mathrm{z.zm}_0{}'[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}'])(\lambda \mathrm{z.zm}_0{}'[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']))\to\to_\beta$
$If(And0'1')\Omega(If\mathrm{m}_0{}'(\lambda \mathrm{z.zm}_0{}'[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}'])(\lambda \mathrm{z.zm}_0{}'[\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']))\to\to_\beta$
$If0'\Omega[\mathrm{m}_0{}',\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']\to\to_\beta[\mathrm{m}_0{}',\mathrm{m}_1{}',\ldots,\mathrm{m}_k{}']$.

(b) $Cons\mathrm{m}'(\mathrm{n}+1)'\equiv$
$(\lambda \mathrm{xy}.If(And(Atom\mathrm{y})(Not(Null\mathrm{y})))\Omega(If\mathrm{x}(\lambda \mathrm{z.zxy})(\lambda \mathrm{z.zxy})))\mathrm{m}'(\mathrm{n}+1)'\to\to_\beta$
$If(And(Atom(\mathrm{n}+1)')(Not(Null(\mathrm{n}+1)')))\Omega(If m'(\lambda \mathrm{z.zm}'(\mathrm{n}+1)')(\lambda \mathrm{z.zm}'(\mathrm{n}+1)'))\to\to_\beta$
$If(And1'1')\Omega(If m'(\lambda \mathrm{z.zm}'(\mathrm{n}+1)')(\lambda \mathrm{z.zm}'(\mathrm{n}+1)'))\to\to_\beta$
$If1'\Omega(If m'(\lambda \mathrm{z.zm}'(\mathrm{n}+1)')(\lambda \mathrm{z.zm}'(\mathrm{n}+1)'))\to\to_\beta\Omega$ and the term $Cons\mathrm{m}'(\mathrm{n}+1)'$ does not have a head normal form.

(c) $Cons\mathrm{m}'\Omega\equiv(\lambda \mathrm{xy}.If(And(Atom\mathrm{y})(Not(Null\mathrm{y})))\Omega(If\mathrm{x}(\lambda \mathrm{z.zxy})(\lambda \mathrm{z.zxy})))\mathrm{m}'\Omega\to\to_\beta$
$If(And(Atom\Omega)(Not(Null\Omega)))\Omega(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\equiv$
$(\lambda \mathrm{xyz}.(Null\mathrm{x})\mathrm{Tyz})(And(Atom\Omega)(Not(Null\Omega)))\Omega(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\to\to_\beta$
$Null(And(Atom\Omega)(Not(Null\Omega)))\mathrm{T}\Omega(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\equiv$
$(\lambda \mathrm{x.x}(\lambda \mathrm{xyz.z})\mathrm{T}0'1')(And(Atom\Omega)(Not(Null\Omega)))\mathrm{T}\Omega(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\to_\beta$
$And(Atom\Omega)(Not(Null\Omega))(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}\Omega(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\equiv$
$(\lambda \mathrm{xy}.If(Null\mathrm{x})\mathrm{xy})(Atom\Omega)(Not(Null\Omega))(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}\Omega$
$(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\to\to_\beta$
$If(Null(Atom\Omega))(Atom\Omega)(Not(Null\Omega))(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}\Omega$
$(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\equiv$
$(\lambda \mathrm{xyz}.(Null\mathrm{x})\mathrm{Tyz})(Null(Atom\Omega))(Atom\Omega)(Not(Null\Omega))(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}\Omega$
$(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\to\to_\beta$
$Null(Null(Atom\Omega))\mathrm{T}(Atom\Omega)(Not(Null\Omega))(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}\Omega$
$(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\equiv$
$(\lambda \mathrm{x.x}(\lambda \mathrm{xyz.z})\mathrm{T}0'1')(Null(Atom\Omega))\mathrm{T}(Atom\Omega)(Not(Null\Omega))(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}\Omega$
$(If m'(\lambda \mathrm{z.zm}'\Omega)(\lambda \mathrm{z.zm}'\Omega))\to_\beta$
$Null(Atom\Omega)(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}(Atom\Omega)(Not(Null\Omega))(\lambda \mathrm{xyz.z})$

$T0'1'T\Omega(\mathit{If}\,m'(\lambda z.zm'\Omega)(\lambda z.zm'\Omega))\equiv$
$(\lambda x.x(\lambda xyz.z)T0'1')(\mathit{Atom}\Omega)(\lambda xyz.z)T0'1'T(\mathit{Atom}\Omega)(\mathit{Not}(\mathit{Null}\Omega))(\lambda xyz.z)$
$T0'1'T\Omega(\mathit{If}\,m'(\lambda z.zm'\Omega)(\lambda z.zm'\Omega))\to_\beta$
$(\mathit{Atom}\Omega)(\lambda xyz.z)T0'1'(\lambda xyz.z)T0'1'T(\mathit{Atom}\Omega)(\mathit{Not}(\mathit{Null}\Omega))(\lambda xyz.z)$
$T0'1'T\Omega(\mathit{If}\,m'(\lambda z.zm'\Omega)(\lambda z.zm'\Omega))$ and, according to Proposition 1, the term $\mathit{Cons}m'\Omega$ does not have a head normal form.

(d) There are two possible cases: d1) k=0, d2) k>0.

(d1) $\mathit{Cons}\Omega[\,]\equiv$
$(\lambda xy.\mathit{If}(\mathit{And}(\mathit{Atom}y)(\mathit{Not}(\mathit{Null}y)))\Omega(\mathit{If}x(\lambda z.zxy)(\lambda z.zxy)))\Omega[\,]\to\to_\beta$
$\mathit{If}(\mathit{And}(\mathit{Atom}[\,])(\mathit{Not}(\mathit{Null}[\,])))\Omega(\mathit{If}\Omega(\lambda z.z\Omega[\,])(\lambda z.z\Omega[\,]))\to\to_\beta$
$\mathit{If}(\mathit{And}1'0')\Omega(\mathit{If}\Omega(\lambda z.z\Omega[\,])(\lambda z.z\Omega[\,]))\to\to_\beta$
$\mathit{If}0'\Omega(\mathit{If}\Omega(\lambda z.z\Omega[\,])(\lambda z.z\Omega[\,]))\to\to_\beta\mathit{If}\Omega(\lambda z.z\Omega[\,])(\lambda z.z\Omega[\,])\equiv$
$(\lambda xyz.(\mathit{Null}x)Tyz)\Omega(\lambda z.z\Omega[\,])(\lambda z.z\Omega[\,])\to\to_\beta$
$(\mathit{Null}\Omega)T(\lambda z.z\Omega[\,])(\lambda z.z\Omega[\,])$ and, according to Proposition 1, the term $\mathit{Cons}\Omega[\,]$ does not have a head normal form.

(d2) $\mathit{Cons}\Omega[m_1',\dots,m_k']\equiv(\lambda xy.\mathit{If}(\mathit{And}(\mathit{Atom}y)(\mathit{Not}(\mathit{Null}y)))\Omega$
$(\mathit{If}x(\lambda z.zxy)(\lambda z.zxy)))\Omega[m_1',\dots,m_k']\to\to_\beta$
$\mathit{If}(\mathit{And}(\mathit{Atom}[m_1',\dots,m_k'])(\mathit{Not}(\mathit{Null}[m_1',\dots,m_k'])))\Omega$
$(\mathit{If}\Omega(\lambda z.z\Omega[m_1',\dots,m_k'])(\lambda z.z\Omega[m_1',\dots,m_k']))\to\to_\beta$
$\mathit{If}(\mathit{And}0'1')\Omega(\mathit{If}\Omega(\lambda z.z\Omega[m_1',\dots,m_k'])(\lambda z.z\Omega[m_1',\dots,m_k']))\to\to_\beta$
$\mathit{If}0'\Omega(\mathit{If}\Omega(\lambda z.z\Omega[m_1',\dots,m_k'])(\lambda z.z\Omega[m_1',\dots,m_k']))\to\to_\beta$
$\mathit{If}\Omega(\lambda z.z\Omega[m_1',\dots,m_k'])(\lambda z.z\Omega[m_1',\dots,m_k'])\equiv$
$(\lambda xyz.(\mathit{Null}x)Tyz)\Omega(\lambda z.z\Omega[m_1',\dots,m_k'])(\lambda z.z\Omega[m_1',\dots,m_k'])\to\to_\beta$
$(\mathit{Null}\Omega)T(\lambda z.z\Omega[m_1',\dots,m_k'])(\lambda z.z\Omega[m_1',\dots,m_k'])$ and, according to Proposition 1, the term $\mathit{Cons}\Omega[m_1',\dots,m_k']$ does not have a head normal form.

We define the auxiliary term $\mathit{Zero}\equiv\lambda x.\mathit{If}(\mathit{Atom}x)(\mathit{Null}x)\Omega$ and show the following:

a) $\mathit{Zero}0'\to\to_\beta 1'$.

b) $\mathit{Zero}(n+1)'\to\to_\beta 0'$.

c) $\mathit{Zero}[m_1',\dots,m_k']$, where $m_i\in$S-expressions, $i=1,\dots,k$, $k\geq 1$, does not have a head normal form.

d) $\mathit{Zero}\Omega$ does not have a head normal form.

(a) $\mathit{Zero}0'\equiv(\lambda x.\mathit{If}(\mathit{Atom}x)(\mathit{Null}x)\Omega)0'\to_\beta\mathit{If}(\mathit{Atom}0')(\mathit{Null}0')\Omega\to\to_\beta$
$\mathit{If}1'1'\Omega\to\to_\beta 1'$.

(b) $\mathit{Zero}(n+1)'\equiv(\lambda x.\mathit{If}(\mathit{Atom}x)(\mathit{Null}x)\Omega)(n+1)'\to_\beta$
$\mathit{If}(\mathit{Atom}(n+1)')(\mathit{Null}(n+1)')\Omega\to\to_\beta\mathit{If}1'0'\Omega\to\to_\beta 0'$.

(c) $\mathit{Zero}[m_1',\dots,m_k']\equiv(\lambda x.\mathit{If}(\mathit{Atom}x)(\mathit{Null}x)\Omega)[m_1',\dots,m_k']\to_\beta$
$\mathit{If}(\mathit{Atom}[m_1',\dots,m_k'])(\mathit{Null}[m_1',\dots,m_k'])\Omega\to\to_\beta\mathit{If}0'0'\Omega\to\to_\beta\Omega$ and the term

$Zero[m_1', \ldots, m_k']$ does not have a head normal form.

(d) $Zero\Omega \equiv (\lambda x.If(Atom x)(Null x)\Omega)\Omega \to_\beta If(Atom\Omega)(Null\Omega)\Omega \equiv$
$(\lambda xyz.(Null x)Tyz)(Atom\Omega)(Null\Omega)\Omega \to\to_\beta Null(Atom\Omega)T(Null\Omega)\Omega \equiv$
$(\lambda x.x(\lambda xyz.z)T0'1')(Atom\Omega)T(Null\Omega)\Omega \to_\beta (Atom\Omega)(\lambda xyz.z)T0'1'T(Null\Omega)\Omega$
and, according to Proposition 1, the term $Zero\Omega$ does not have a head normal form.

Consider the equation:

$$f=\lambda xy.If(Zero x)(If(Zero y)1'0')(If(Zero y)0'(f(xF)(yF))),$$

where $f,x,y \in V$. According to Proposition 3, the term

$$Eq \equiv Y(\lambda fxy.If(Zero x)(If(Zero y)1'0')(If(Zero y)0'(f(xF)(yF))))$$

is a solution of this equation and

$$Eq =_\beta \lambda xy.If(Zero x)(If(Zero y)1'0')(If(Zero y)0'(Eq(xF)(yF))).$$

According to Proposition 2, we have:

$$Eq m_1' m_2' =_\beta (\lambda xy.If(Zero x)(If(Zero y)1'0')(If(Zero y)0'(Eq(xF)(yF))))m_1' m_2',$$

where $m_1, m_2 \in M$.

Let us show that the term Eq λ-defines the function eq. To do this, show the following:

a) $Eq n_1' n_2' =_\beta 1'$, where $n_1, n_2 \in N$ and $n_1 = n_2$; and $Eq n_1' n_2' =_\beta 0'$, where $n_1, n_2 \in N$ and $n_1 \neq n_2$.

b) $Eq m_1' m_2'$, where $m_1 \in M$ and $m_1 \notin N$, or $m_2 \in M$ and $m_2 \notin N$, does not have a head normal form.

(a) The proof is by induction on n_2.
Let $n_2 = 0$, we show that i) $Eq 0'0' =_\beta 1'$ and ii) $Eq n_1' 0' =_\beta 0'$ if $n_1 > 0$.

(i) $Eq 0'0' =_\beta$
$(\lambda xy.If(Zero x)(If(Zero y)1'0')(If(Zero y)0'(Eq(xF)(yF))))0'0' \to\to_\beta$
$If(Zero 0')(If(Zero 0')1'0')(If(Zero 0')0'(Eq(0'F)(0'F))) \to\to_\beta$
$If 1'(If 1'1'0')(If(Zero 0')0'(Eq(0'F)(0'F))) \to\to_\beta If 1'1'0' \to\to_\beta 1'$.

(ii) $Eq n_1' 0' =_\beta$
$(\lambda xy.If(Zero x)(If(Zero y)1'0')(If(Zero y)0'(Eq(xF)(yF))))n_1' 0' \to\to_\beta$
$If(Zero n_1')(If(Zero 0')1'0')(If(Zero 0')0'(Eq(n_1' F)(0'F))) \to\to_\beta$
$If 0'(If(Zero 0')1'0')(If(Zero 0')0'(Eq(n_1' F)(0'F))) \to\to_\beta$
$If(Zero 0')0'(Eq(n_1' F)(0'F)) \to\to_\beta If 1'0'(Eq(n_1' F)(0'F)) \to\to_\beta 0'$.

Let $n_2 > 0$. Assuming that the assertion is true for $n_2 - 1$, we prove it for n_2. We show that i) $Eq 0' n_2' =_\beta 0'$ and ii) $Eq n_1' n_2' =_\beta 0'$ if $n_1 > 0$ and $n_1 \neq n_2$; $Eq n_1' n_2' =_\beta 1'$ if $n_1 > 0$ and $n_1 = n_2$.

(i) $Eq0'n_2' =_\beta$
$(\lambda xy.If(Zero\,x)(If(Zero\,y)1'0')(If(Zero\,y)0'(Eq(xF)(yF))))0'n_2' \to\to_\beta$
$If(Zero0')(If(Zero\,n_2')1'0')(If(Zero\,n_2')0'(Eq(0'F)(n_2'F))) \to\to_\beta$
$If1'(If0'1'0')(If(Zero\,n_2')0'(Eq(0'F)(n_2'F))) \to\to_\beta If0'1'0' \to\to_\beta 0'$.

(ii) $Eq\,n_1'n_2' =_\beta$
$(\lambda xy.If(Zero\,x)(If(Zero\,y)1'0')(If(Zero\,y)0'(Eq(xF)(yF))))n_1'n_2' \to\to_\beta$
$If(Zero\,n_1')(If(Zero\,n_2')1'0')(If(Zero\,n_2')0'(Eq(n_1'F)(n_2'F))) \to\to_\beta$
$If0'(If0'1'0')(If(Zero\,n_2')0'(Eq(n_1'F)(n_2'F))) \to\to_\beta$
$If(Zero\,n_2')0'(Eq(n_1'F)(n_2'F)) \to\to_\beta If0'0'(Eq(n_1'F)(n_2'F)) \to\to_\beta$
$Eq(n_1'F)(n_2'F) \equiv Eq((\lambda x.xF(n_1-1)')F)((\lambda x.xF(n_2-1)')F) \to\to_\beta$
$Eq(FF(n_1-1)')(FF(n_2-1)') \to\to_\beta Eq(n_1-1)'(n_2-1)'$, further we have:
if $n_1 \neq n_2$, then, by the induction hypothesis, we have: $Eq(n_1-1)'(n_2-1)' =_\beta 0'$;
if $n_1 = n_2$, then, by the induction hypothesis, we have: $Eq(n_1-1)'(n_2-1)' =_\beta 1'$.

(b) There are two possible cases: b1) $m_1 \in M$ and $m_1 \notin N$, b2) $m_1 \in N$, $m_2 \in M$ and $m_2 \notin N$.

(b1) $Eq\,m_1'm_2' =_\beta$
$(\lambda xy.If(Zero\,x)(If(Zero\,y)1'0')(If(Zero\,y)0'(Eq(xF)(yF))))m_1'm_2' \to\to_\beta$
$If(Zero\,m_1')(If(Zero\,m_2')1'0')(If(Zero\,m_2')0'(Eq(m_1'F)(m_2'F))) \equiv$
$(\lambda xyz.(Null\,x)Tyz)(Zero\,m_1')(If(Zero\,m_2')1'0')$
$(If(Zero\,m_2')0'(Eq(m_1'F)(m_2'F))) \to\to_\beta$
$Null(Zero\,m_1')T(If(Zero\,m_2')1'0')(If(Zero\,m_2')0'(Eq(m_1'F)(m_2'F))) \equiv$
$(\lambda x.x(\lambda xyz.z)T0'1')(Zero\,m_1')T(If(Zero\,m_2')1'0')$
$(If(Zero\,m_2')0'(Eq(m_1'F)(m_2'F))) \to_\beta$
$(Zero\,m_1')(\lambda xyz.z)T0'1'T(If(Zero\,m_2')1'0')(If(Zero\,m_2')0'(Eq(m_1'F)(m_2'F)))$
and, according to Proposition 1, the term $Eq\,m_1'm_2'$ does not have a head normal form.

(b2) We show that i) $Eq0'm_2'$ and ii) $Eq(n+1)'m_2'$, where $n \in N$, do not have a head normal form.

(i) $Eq0'm_2' =_\beta$
$(\lambda xy.If(Zero\,x)(If(Zero\,y)1'0')(If(Zero\,y)0'(Eq(xF)(yF))))0'm_2' \to\to_\beta$
$If(Zero0')(If(Zero\,m_2')1'0')(If(Zero\,m_2')0'(Eq(0'F)(m_2'F))) \to\to_\beta$
$If1'(If(Zero\,m_2')1'0')(If(Zero\,m_2')0'(Eq(0'F)(m_2'F))) \to\to_\beta$
$If(Zero\,m_2')1'0' \equiv (\lambda xyz.(Null\,x)Tyz)(Zero\,m_2')1'0' \to\to_\beta$
$Null(Zero\,m_2')T1'0' \equiv (\lambda x.x(\lambda xyz.z)T0'1')(Zero\,m_2')T1'0' \to_\beta$
$(Zero\,m_2')(\lambda xyz.z)T0'1'T1'0'$ and, according to Proposition 1, the term $Eq0'm_2'$ does not have a head normal form.

(ii) $Eq(n+1)'m_2' =_\beta$
$(\lambda xy.If(Zero\,x)(If(Zero\,y)1'0')(If(Zero\,y)0'(Eq(xF)(yF))))(n+1)'m_2' \to\to_\beta$
$If(Zero(n+1)')(If(Zero\,m_2')1'0')(If(Zero\,m_2')0'(Eq((n+1)'F)(m_2'F))) \to\to_\beta$
$If0'(If(Zero\,m_2')1'0')(If(Zero\,m_2')0'(Eq((n+1)'F)(m_2'F))) \to\to_\beta$
$If(Zero\,m_2')0'(Eq((n+1)'F)(m_2'F)) \equiv$

$(\lambda \mathrm{xyz}.(\mathit{Null}\mathrm{x})\mathrm{Tyz})(\mathit{Zero}\mathrm{m}_2{}')0'(\mathit{Eq}((\mathrm{n+1})'\mathrm{F})(\mathrm{m}_2{}'\mathrm{F}))\to\to_\beta$
$\mathit{Null}(\mathit{Zero}\mathrm{m}_2{}')\mathrm{T}0'(\mathit{Eq}((\mathrm{n+1})'\mathrm{F})(\mathrm{m}_2{}'\mathrm{F}))\equiv$
$(\lambda \mathrm{x.x}(\lambda \mathrm{xyz.z})\mathrm{T}0'1')(\mathit{Zero}\mathrm{m}_2{}')\mathrm{T}0'(\mathit{Eq}((\mathrm{n+1})'\mathrm{F})(\mathrm{m}_2{}'\mathrm{F}))\to_\beta$
$(\mathit{Zero}\mathrm{m}_2{}')(\lambda \mathrm{xyz.z})\mathrm{T}0'1'\mathrm{T}0'(\mathit{Eq}((\mathrm{n+1})'\mathrm{F})(\mathrm{m}_2{}'\mathrm{F}))$ and, according to Proposition 1, the term $\mathit{Eq}(\mathrm{n+1})'\mathrm{m}_2{}'$ does not have a head normal form.

□

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ՆԵՐԴՐՎԱԾ ՄԱԿԿԱՐՏԻԻ ՖՈՒՆԿՑԻԱՆԵՐԻ λ-ՈՐՈՇԵԼԻՈՒԹՅՈՒՆԸ, ՈՐՊԵՍ ԱՐԳՈՒՄԵՆՏՆԵՐԻ ԱՆՈՐՈՇ ԱՐԺԵՔՆԵՐՈՎ ՖՈՒՆԿՑԻԱՆԵՐ

Ծրագրավորման լեզուների ներդրված ֆունկցիաները հանդիսանում են արգումենտների անորոշ արժեքներով ֆունկցիաներ: *car*, *cdr*, *cons*, *null*, *atom*, *if*, *eq*, *not*, *and*, *or* ներդրված Մակկարտիի ֆունկցիաները օգտագործվում են բոլոր ֆունկցիոնալ ծրագրավորման լեզուներում: Այս աշխատանքում ապացուցված է ներդրված Մակկարտիի ֆունկցիաների λ-որոշելիությունը, որպես արգումենտների անորոշ արժեքներով ֆունկցիաներ: Այս արդյունքն անհրաժեշտ է տիպիզացված ֆունկցիոնալ ծրագրավորման լեզուները ոչ տիպիզացված ֆունկցիոնալ ծրագրավորման լեզուների թարգմանության ժամանակ: