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Mathematics

CONDITIONAL MOMENTS OF THE DISTANCE DISTRIBUTION TWO RANDOM POINTS IN A CONVEX DOMAIN IN $\mathbf{R}^2$

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In this article we define two new integral geometric concepts: conditional moments of the chord length distribution of a convex domain and conditional moments of the distance distribution of two independent uniformly distributed points in a convex domain. We also found a relation between these two concepts.

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Introduction. Today, tomography is one of the rapidly developing areas of the mathematics. Geometric tomography (the term introduced by R. Gardner in [1]) is a field of mathematics engaged in extracting information about a geometric object from data on its sections or projections to reconstruct the geometric object. The reconstruction of a convex domain using random sections makes it possible to simplify the calculation, since mathematical statistics methods can be used to estimate the geometric characteristics of random sections. The integral geometric concepts such as the distribution of the chord length, the distribution of the distance between two random points in a convex domain D and many others carry some information about D. In this article we define two new integral geometric concepts: conditional moments of the chord length distribution of a convex domain and conditional moments of the distribution of the distance of two random points in a convex domain in $\mathbf{R}^2$. We also find the relation between these two concepts.

By $\mathbf{R}^2$ we denote the Euclidean plane, by $\mathbf{S}^1$ the unit circle in the plane centered at the origin. We denote by G the space of lines in the plane, and by $\mathcal{N}$ the set of nonnegative integers. We use the usual parametrization of a line $g \in G$, $g = (p, \varphi)$: p is the distance of G from the origin O; $\varphi \in \mathbf{S}^1$ is the direction normal to G.

It is well known (see [2]) that the invariant measure dg can be decomposed

$$dg = dp \cdot d\varphi,$$

where $d\varphi$ is the element of the arc measure on $\mathbf{S}^1$.

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Definition 1. *A compact convex set $D \subset \mathbf{R}^2$ well be called a convex domain. We consider the random line g with normalized invariant measure $\left(\frac{dg}{L}\right.$, here L is the perimeter of the domain $\left.\right)$ and we denote the length of the chord $D \cap g$ by $X(g)$. The conditional n-th moment of the distribution of the chord length (with respect to condition $X > u$) we define as*

$$I_{n,u} = \frac{1}{L} \int\limits_{X(g)>u} X(g)^n dg. \tag{1}$$

Also by $F_X(t)$ we denote the distribution function of $X(g)$.

Definition 2. *For two independent uniformly distributed points P_1, P_2 in a convex domain D we denote the distance between the points by $r = |P_1 - P_2|$. The conditional n-th moment of the distribution of the distance (with respect to condition $r > u$) we define as*

$$J_{n,u} = \frac{1}{S^2} \int\limits_{|P_1-P_2|>u} r^n dP_1 dP_2, \tag{2}$$

where S is the area of D, dP_i $(i = 1,2)$ is the usual Lebesgue measure in $\mathbf{R}^2$. By $F_r(u)$ we denote the distribution function of the distance of two uniformly distributed points P_1, P_2 in a convex domain D.

The moments of the distribution of the chord length and the distribution of the distance between two independent uniformly distributed points in a convex domain was considered in [2].

Theorem 1. *Let D be a convex domain and $u \geq 0$. For any $n \in \mathcal{N}$ the relation between the conditional moments of the distance distribution of two random points in D and the conditional moments of the distribution of the chord length is*

$$J_{n,u} = \frac{2L}{S^2}\left(\frac{I_{0,u}u^{n+3}}{n+3} - \frac{I_{1,u}u^{n+2}}{n+2} + \frac{I_{n+3,u}}{(n+2)(n+3)}\right). \tag{3}$$

The Proof of Theorem 1. Given pair of points (P_1, P_2) in the plane. There are two equivalent representations of the pair.

1. A pair of points (P_1, P_2) can be determined by the usual Cartesian coordinates

$$(P_1, P_2) = (x_1, y_1, x_2, y_2).$$

2. A pair of points (P_1, P_2) can be determined by the line $g = (p, \varphi)$ passing through the points and two one dimensional coordinates (t_1, t_2), which determine P_1 and P_2 on the line g. Thus

$$(P_1, P_2) = (g, t_1, t_2) = (p, \varphi, t_1, t_2).$$

Note that as a reference point on g one can take the perpendicular of the origin O onto g.

It is well-known that (see [2])

$$dP_1 dP_2 = |t_2 - t_1|\, dg dt_1 dt_2.$$

We consider two points t_1 and t_2 chosen at random, independently and with a uniform distribution on a segment of length X.

To prove Theorem 1, we need to prove the following lemma.

Lemma **.** *For $u \le X$ and $n \in \mathbb{N}$ we have*

$$\int\limits_{|t_1-t_2|>u} r^{n+1} dt_1 dt_2 = 2\left(\frac{u^{n+3}}{n+3} - \frac{Xu^{n+2}}{n+2} + \frac{X^{n+3}}{(n+2)(n+3)}\right). \tag{4}$$

Proof. By symmetry we have

$$\int\limits_{|t_1-t_2|>u} r^{n+1} dt_1 dt_2 = 2\int\limits_0^{X-u} dt_1 \int\limits_{t_1+u}^{X} (t_2-t_1)^{n+1} dt_2 =$$

$$= 2\int\limits_0^{X-u} \frac{(X-t_1)^{n+2}}{n+2} - \frac{u^{n+2}}{n+2} dt_1 = 2\left(\frac{u^{n+3}}{n+3} - \frac{Xu^{n+2}}{n+2} + \frac{X^{n+3}}{(n+2)(n+3)}\right). \tag{5}$$

Proof of Theorem 1. For two points P_1 and P_2 we consider the line g passing through the points. Using (5) and taking into account (4), we have

$$J_{n,u} = \frac{1}{S^2} \int\limits_{|P_1-P_2|>u} r^n dP_1 dP_2 = \frac{1}{S^2} \int\limits_{X(g)>u} \int\limits_{|t_1-t_2|>u} r^{n+1} dt_1 dt_2 dg =$$

$$= \frac{2}{S^2} \int\limits_{X(g)>u} \left(\frac{u^{n+3}}{n+3} - \frac{X(g)u^{n+2}}{n+2} + \frac{X(g)^{n+3}}{(n+2)(n+3)}\right) dg =$$

$$= \frac{2L}{S^2}\left(\frac{I_{0,u}u^{n+3}}{n+3} - \frac{I_{1,u}u^{n+2}}{n+2} + \frac{I_{n+3,u}}{(n+2)(n+3)}\right). \tag{6}$$

It is obvious that when $u \ge Diam(D)$, both sides are 0.

Thus we obtained for all $u \ge 0$.

Corollary 1. *For $u = 0$ we get*

$$J_{n,0} = \frac{2L}{S^2}\left(\frac{I_{n+3,0}}{(n+2)(n+3)}\right). \tag{7}$$

Note that (7) was found in [2].

Corollary 2. *For $n = 0$ we get*

$$J_{0,u} = \frac{2L}{S^2}\left(\frac{I_{0,u}u^3}{3} - \frac{I_{1,u}u^2}{2} + \frac{I_{3,u}}{6}\right). \tag{8}$$

Taking into account

$$J_{0,u} = \frac{1}{S^2}\int_{|P_1-P_2|>u} dP_1 dP_2 = 1 - F_r(u), \tag{9}$$

we get the following theorem.

For the distribution function of the distance between two independent uniformly distributed points in a convex domain we will have

Theorem 2.

$$F_r(u) = 1 - J_{0,u} = 1 - \frac{2L}{S^2}\left(\frac{I_{0,u}u^3}{3} - \frac{I_{1,u}u^2}{2} + \frac{I_{3,u}}{6}\right). \tag{10}$$

Representation for $I_{n,u}$. Now we are going to find representations for $I_{0,u}, I_{1,u}, I_{3,u}$.

1. For $I_{0,u}$ we have

$$I_{0,u} = \frac{1}{L} \int\limits_{X(g)>u} dg = P(X(g) > u) = (1 - P(X(g) \le u)) = 1 - F_X(u). \quad (11)$$

Here $F_X(t)$ is the chord length distribution a function.

2. For the derivative of $I_{1,u}$ we have

$$(I_{1,u})' = \frac{1}{L} \left(\int\limits_{X(g)>u} X(g)\, dg \right)' = -\lim_{\Delta u \to 0} \frac{\frac{1}{L} \int\limits_{u<X(g)<u+\Delta u} X(g)\, dg}{\Delta u} =$$

$$-\lim_{\Delta u \to 0} \frac{uP(u < X(g) < u + \Delta u)}{\Delta u} = -u f_X(u), \quad (12)$$

where $f_X(t)$ is the density function of the chord length distribution of $X(g)$.

Integrating (12) and taking into account that $I_{1,0} = \frac{\pi S}{L}$, we get (see in [3]):

$$I_{1,u} = \frac{\pi S}{L} - \int\limits_0^u v f_x(v)\, dv. \quad (13)$$

We can see it in [4].

3. For the derivative of $I_{3,u}$ we have

$$(I_{3,u})' = \frac{1}{L} \left(\int\limits_{X(g)>u} X(g)^3\, dg \right)' = -\lim_{\Delta u \to 0} \frac{\frac{1}{L} \int\limits_{u<X(g)<u+\Delta u} X(g)^3\, dg}{\Delta u} =$$

$$-\lim_{\Delta u \to 0} \frac{u^3 P(u < X(g) < u + \Delta u)}{\Delta u} = -u^3 f_X(u). \quad (14)$$

Integrating (14) and taking into account that $I_{3,0} = \frac{3S^2}{L}$ (see [2]), we get

$$I_{3,u} = \frac{3S^2}{L} - \int\limits_0^u v^3 f_x(v)\, dv. \quad (15)$$

Finally, substituting (11), (13), (15) into (17), we obtain the following theorem.

Theorem 3. *Let D be a convex domain and $u \ge 0$. The following relation between the distribution function of the distance of two uniformly distributed points of D and the chord length distribution function of D is valid:*

$$F_r(u) = 1 - \frac{L}{S^2} \left(\frac{2u^3}{3} - \frac{\pi S u^2}{L} + \frac{S^2}{L} - u^2 \int\limits_0^u F_x(v)\, dv + \int\limits_0^u v^2 F_x(v)\, dv \right). \quad (16)$$

For the density function $f_r(u)$ of the distance of two uniformly distributed points of D we obtain

$$f_r(u) = \frac{1}{S^2}\left(L\left(2u\int_0^u F_x(v)\,dv + u^2F_x(u) - u^2F_x(u)\right) - 2Lu^2 + 2\pi Su\right) =$$

$$= \frac{1}{S^2}\left(2Lu\left(\int_0^u F_x(v)\,dv\right) - 2Lu^2 + 2\pi Su\right). \quad (17)$$

Note that formula (17) was obtained in [4] and [5].

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ԵՐԿՉԱՓ ՈՒՌՈՒՑԻԿ ՏԻՐՈՒՅԹՈՒՄ $\mathbf{R}^2$ ԵՐԿՈՒ ՊԱՏԱՀԱԿԱՆ ԿԵՏԵՐԻ ՄԻՋԵՎ ՀԵՌԱՎՈՐՈՒԹՅԱՆ ԲԱՇԽՄԱՆ ՊԱՅՄԱՆԱԿԱՆ ՄՈՄԵՆՏՆԵՐ

Այս հոդվացում մենք սահմանում ենք ինտեգրալ երկրաչափության երկու նոր հասկացություններ՝ ուռուցիկ մարմնի պապահական լարի երկարության պայմանական մոմենտներ և ուռուցիկ մարմնում երկու անկախ հավասարաչափ բաշխված կետերի հեռավորության միջև պայմանական մոմենտներ: Հոդվածում, նաև գտնված են այս երկու հասկացությունների միջև կապը:

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УСЛОВНЫЕ МОМЕНТЫ РАСПРЕДЕЛЕНИЯ РАССТОЯНИЯ МЕЖДУ ДВУМЯ СЛУЧАЙНЫМИ ТОЧКАМИ В ВЫПУКЛОЙ ОБЛАСТИ $\mathbf{R}^2$

В этой статье мы определяем два новых интегрально-геометрических понятия: условные моменты распределения длины хорды выпуклой области и условные моменты распределения расстояния между двумя независимыми равномерно распределенными точками в выпуклой области. Также в этой статье найдена связь между этими двумя понятиями.