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Mathematics

ON THE UNCONDITIONAL CONVERGENCE OF FABER–SCHAUDER SERIES IN L^1

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In this paper we proved that the Faber–Schauder functions form an unconditional representation system for L^1.

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Introduction. A basis of a Banach space X, is a countable set $B = \{x_n \in X : n \in \mathbb{N}\}$ such that each $x \in X$ can be uniquely represented by series $\sum\limits_{n=1}^{\infty} A_n(x)x_n$ converging to x in the norm of X. If $\sum\limits_{n=1}^{\infty} A_n(x)x_n$ converges after any rearrangement of the terms, then the series is an unconditional representation of x, and the basis is called unconditional basis.

Let E be a measurable set with positive measure, and let S be a metric space of measurable functions $f(x)$, $x \in E$.

Definition. *A system $\{g_n(x)\}$, $g_n(x) \in S, n = 1,2,\dots,$ is called system of unconditional representation for the space S, if for every $f \in S$ there is a series $\sum\limits_{n=0}^{\infty} b_n g_n(x)$, which converges unconditionally to f in the metric of the space S, that is for any rearrangement $\{\pi(n)\}$ of the natural numbers the series $\sum\limits_{n=0}^{\infty} b_{\pi(n)} g_{\pi(n)}(x)$ converges to f in the metric of S.*

The basisness of the Faber–Schauder system in $C[0,1]$ (see [1]) provides variety of representation theorems. An example of such result is Talalyan's theorem [2] (see also [3]) namely, for each measurable function on $[0,1]$ there exists a Faber–Schauder series with coefficients converging to zero that converges to the function almost everywhere. This is an analogue of (Luzin's [4]) Menchoff's [5] theorem for

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the trigonometric system. Note that these expansions do not converge unconditionally, and it is known that there is no unconditional basis for $L[0,1]$ or $C[0,1]$ (see [6]). Nevertheless, in [7] it is proved that for every $\varepsilon \in (0,1)$ there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1-\varepsilon$ such that for every function $f(x) \subset [0,1]$ there is a series with respect to Faber–Schauder system, which unconditionally converges to $f(x)$ on E.

It should be noted that this is a sharp result, since the set E in the statement cannot be replaced by $[0,1]$. Here $|E|$ is a Lebesgue measure of E. There are also a lot of results connected with Faber–Schauder system [8–12].

Since there is no unconditional basis in L^1, Faber–Schauder system is not an unconditional basis in L^1. In this paper we will prove that the Faber–Schauder system is an unconditional representation system for $L[0,1]$. Moreover, the following theorem is true.

Theorem. *For any natural number m_0 and for each $f \in L[0,1)$ there exists a Faber–Schauder series $\sum_{n=m_0}^{\infty} b_n \varphi_n(x)$, with coefficients converging to zero, which converges unconditionally to f in the norm of $L[0,1)$.*

It is easy to see that this theorem is not true for the other classical (trigonometric, Walsh, Haar, Franklin ...) systems.

The functions of the Faber–Schauder system, $\Phi = \{\varphi_n \colon n = 0,1,\dots\}$, are the continuous, piecewise–linear functions on $[0,1]$, given by $\varphi_0(x) = 1$, $\varphi_1(x) = x$, and for $n = 2^k + i$, $k = 0,1,\dots;\ i = 1,\dots,2^k$, we have

$$\varphi_n(x) := \varphi_k^{(i)}(x) = \begin{cases} 0, & \text{if } x \notin \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right), \\ 1, & \text{if } x = x_n = x_k^{(i)} = \frac{2i-1}{2^{k+1}}, \end{cases}$$

and is linear and continuous on the intervals $\left[\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}\right]$, $\left[\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}\right]$. The corresponding linear functionals are given by

$$A_0(f) = f(0),\ A_1(f) = f(1) - f(0),$$

and for $n > 1$

$$A_n(f) = A_{k,i}(f) = f\left(\frac{2i-1}{2^{k+1}}\right) - \frac{1}{2}\left[f\left(\frac{i-1}{2^k}\right) + f\left(\frac{i}{2^k}\right)\right].$$

Recall that the Faber–Schauder system is a basis for the space $C[0,1]$ (see [1]). Moreover, for each function $f(x) \in C[0,1]$ the series

$$\sum_{n=0}^{\infty} A_n(f)\varphi_n(x),$$

converges uniformly to f on $[0,1]$. For a set E we denote its characteristic function by $\chi_E(x)$.

$$\chi_E(x) := \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

We denote the support of the function $\varphi_n(x)=\varphi_k^{(i)}(x)$ by $\Delta_n=\Delta_k^{(i)}$. We will consider functions of the form $f=\sum_{\nu=1}^{2^p}\gamma_\nu\chi_{[\frac{\nu-1}{2^p},\frac{\nu}{2^p})}$ dyadic step-functions of rank p.

As we know there are functions in $C[0,1]$ that cannot be represented by Faber–Schauder series converging unconditionally in $C[0,1]$.

The proof of the Theorem is based on a proper approximation of the characteristic functions of dyadic intervals by Faber–Schauder polynomials of high rank.

Auxiliary Lemmas.

Lemma 1. *Let* $\Delta=\left[\frac{i-1}{2^p},\frac{i}{2^p}\right)$, $\gamma\neq 0$, $\varepsilon\in(0,1)$, *and* N_0 *be a natural number. There exists a Faber–Schauder polynomial*

$$Q(x)=\sum_{n=N_0}^{N}A_n\varphi_n(x)$$

such that

$$|A_n|\leq|\gamma|,\forall n\in[N_0,N],$$

$$\int_0^1\left|Q(x)-\gamma\chi_{\Delta(x)}\right|dx<\varepsilon,$$

$$\sum_{n=N_0}^{N}|A_n|\varphi_n(x)=0,\ \textit{if}\ x\in[0,1]\setminus\Delta,$$

$$\int_0^1\left|\sum_{n=N_0}^{N}|A_n|\varphi_n(x)\right|dx<2|\gamma||\Delta|.$$

Proof. Assume, without loss of generality, that $N_0>2^p+i$. Then some of the dyadic points x_n, with $n<N_0$, lie in Δ. Denote those points by $x_{n_1},x_{n_2},\dots,x_{n_\ell}$, one choose $q\in\mathbb{N}$ such that $q>\log_2\left(|\gamma|(\ell+1)/\varepsilon\right)+1$, and let

$$E=\Delta\setminus\left[\bigcup_{i=1}^{\ell}\left(x_{n_j}-\frac{1}{2^q},x_{n_j}+\frac{1}{2^q}\right)\cup\left(\frac{i-1}{2^p},\frac{i-1}{2^p}+\frac{1}{2^q}\right)\cup\left(\frac{i}{2^p}-\frac{1}{2^q},\frac{i}{2^p}\right)\right].$$

Define the continuous function g by

$$g(x)=\begin{cases}\gamma, & \text{if } x\in E,\\ 0, & \text{if } x\in([0,1]\setminus\Delta)\cup\{x_{n_j};1\leq j\leq\ell\},\end{cases}$$

supposing that g is linear on each of the intervals $\left[x_{n_j}-\frac{1}{2^q},x_{n_j}\right]$, $\left[x_{n_j},x_{n_j}+\frac{1}{2^q}\right]$ for $1\leq j\leq\ell$, $\left[\frac{i-1}{2^p},\frac{i-1}{2^p}+\frac{1}{2^q}\right]$ and $\left[\frac{i}{2^p}-\frac{1}{2^q},\frac{i}{2^p}\right]$.

Indeed we have $2^q>N_0$, $\max(g(x))=|\gamma|$ and $|E|>|\Delta|-\frac{\varepsilon}{2|\gamma|}$.

The Faber–Schauder expansion $g(x) = \sum A_n \varphi_n(x)$ is the required polynomial, $A_n = 0$, if $n < N_0$, or $n > 2^q$. If $N_0 \le n \le 2^q$ and $\Delta_n \subseteq \Delta$, then we have either $A_n = \dfrac{\gamma}{2}$ or γ. Therefore,

$$g(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x) := Q(x),\ N = 2^q,$$

and

$$|A_n| \le |\gamma|,\ A_n\gamma \ge 0,\ \forall n \in [N_0, N].$$

It is not hard to see that

$$\int_0^1 \left|Q(x) - \gamma\chi_{\Delta(x)}\right| dx = \int_\Delta |Q(x) - \gamma|\, dx < 2 \int_{\Delta\setminus E} |\gamma|\, dx \le \varepsilon,$$

$$\int_\Delta \left(\sum_{n=N_0}^{N} |A_n|\varphi_n(x)\right) dx < 2|\gamma||\Delta|$$

and

$$\sum_{n=N_0}^{N} |A_n|\varphi_n(x) = 0,\ \text{if } x \in [0,1]\setminus\Delta.$$

□

Lemma 2. *Let $\varepsilon \in (0,1)$ and N_0 is a natural number, then for each real step function of the form $f = \sum_{\nu=1}^{2^p} \gamma_\nu \chi_{\Delta_\nu}$ where $\gamma_\nu \ne 0$ and $\Delta_\nu = \left[\dfrac{\nu-1}{2^p}, \dfrac{\nu}{2^p}\right) : 1 \le \nu \le 2^p$ is the dyadic partition of $[0,1]$ of rank p, there is a Faber–Schauder polynomial*

$$Q(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x)$$

such that

$$|A_n| \le \varepsilon,\ \forall n \in [N_0, N],$$

$$\int_0^1 |Q(x) - f(x)|\, dx < \varepsilon,$$

and, for each $B \subset \{N_0, \ldots, N\}$,

$$\int_0^1 \left|\sum_{n\in B} A_n \varphi_n(x)\right| dx \le \int_0^1 \left(\sum_{n=N_0}^{N} |A_n|\varphi_n(x)\right) dx \le 2\int_0^1 |f(x)|dx.$$

Proof. We take μ_0 natural number such that

$$\frac{\max\limits_{1\le\nu\le 2^p}(\gamma_\nu)}{\mu_0} \le \varepsilon.$$

We can represent the f function in the form

$$f=\sum_{\nu=1}^{2^p}\gamma_\nu\chi_{\Delta_\nu}=\sum_{k=1}^{\mu_0 2^p}\beta_k\chi_{\widetilde{\Delta}_k},$$

where $\widetilde{\Delta}_k=\Delta_\nu,\ \beta_k=\dfrac{\gamma_\nu}{\mu_0}$ for $k\in[(\nu-1)\mu_0+1,\nu\mu_0]$ and $\nu=1,2,\dots,2^p$.

Successively applying Lemma 1, we get a sequence of Faber–Schauder polynomials $\left\{Q_k(x)\right\}_{k=1}^{\mu_0 2^p}$:

$$Q_k(x)=\sum_{n=N_{k-1}}^{N_k-1}A_n\varphi_n(x),\ N_{k+1}>N_k$$

for all $1\le k\le\mu_0 2^p$, satisfying the following conditions:

$$|A_n|\le|\beta_n|\le\varepsilon,\ \forall n\in[N_{k-1},N_k-1],$$

$$\int_0^1\left|Q_k(x)-\beta_k\chi_{\widetilde{\Delta}_k}(x)\right|dx<\frac{\varepsilon}{\mu_0 2^p},$$

$$\sum_{n=N_{k-1}}^{N_k-1}|A_n|\varphi_n(x)=0,\ \text{if } x\in[0,1]\setminus\widetilde{\Delta}_k,$$

$$\int_0^1\left(\sum_{n=N_{k-1}}^{N_k-1}|A_n|\varphi_n(x)\right)dx<2|\beta_k||\widetilde{\Delta}_k|.$$

Setting

$$Q(x)=\sum_{k=1}^{\mu_0 2^p}Q_k(x)=\sum_{k=1}^{\mu_0 2^p}\sum_{n=N_{k-1}}^{N_k-1}A_n\varphi_n(x)=\sum_{n=N_0}^{N}A_n\varphi_n(x),$$

one has

$$|A_n|\le\varepsilon,\ \forall n\in[N_0,N],$$

$$\int_0^1|Q(x)-f(x)|\,dx\le\sum_{k=1}^{\mu_0 2^p}\int_0^1\left|Q_k(x)-\beta_k\chi_{\widetilde{\Delta}_k}(x)\right|dx<\varepsilon,$$

$$\int_0^1\left(\sum_{n=N_0}^{N}|A_n|\varphi_n(x)\right)dx=\sum_{k=1}^{\mu_0 2^p}\int_{\widetilde{\Delta}_k}\left(\sum_{n=N_{k-1}}^{N_k-1}|A_n|\varphi_n(x)\right)dx$$

$$\le\sum_{k=1}^{\mu_0 2^p}2|\beta_k||\widetilde{\Delta}_k|=2\sum_{\nu=1}^{2^p}\left(\sum_{k=(\nu-1)\mu_0+1}^{\nu\mu_0}|\beta_k|\,|\widetilde{\Delta}_k|\right)=2\sum_{\nu=1}^{2^p}|\gamma_\nu||\Delta_\nu|=2\int_0^1|f(x)|\,dx.$$

□

Proof of the Theorem.

Proof. Let m_0 be a natural number and $f(x) \in L[0,1]$.
It is easy to see that there exist f_1 dyadic step-function such that

$$\|f - f_1\| = \int_0^1 |f(x) - f_1(x)|\, dx < 2^{-2}.$$

By virtue of Lemma 2, there is a Faber–Schauder polynomial

$$Q_1(x) = \sum_{n=m_0}^{m_1-1} A_n \varphi_n(x)$$

such that

$$|A_n| < 2^{-2},\ \forall n \in [m_0, m_1),$$

$$\|Q_1 - f_1\| \le 2^{-2},$$

and for each $B_1 \subset \{m_0, \ldots, m_1 - 1\}$,

$$\left\| \sum_{n \in B_1} A_n \varphi_n(x) \right\| \le 2\|f_1\|.$$

Let the dyadic step-function f_2 satisfy

$$\|(f - Q_1) - f_2\| < 2^{-4},$$

and again apply Lemma 2. We get a Faber–Schauder polynomial

$$Q_2(x) = \sum_{n=m_1}^{m_2-1} A_n \varphi_n(x)$$

such that

$$|A_n| < 2^{-4},\ \forall n \in [m_1, m_2),$$

$$\|Q_2 - f_2\| \le 2^{-4},$$

and, for each $B_2 \subset \{m_1, \ldots, m_2 - 1\}$,

$$\left\| \sum_{n \in B_2} A_n \varphi_n(x) \right\| \le 2\|f_2\|.$$

Then

$$\|f - (Q_1 + Q_2)\| \le 2^{-3}$$

and, since

$$\|f_2\| \le \frac{3}{2^3},$$

we obtain

$$\left\| \sum_{n \in B_2} A_n \varphi_n(x) \right\| \le \frac{3}{2^2}.$$

Continuing this process, one determines a sequence $\{Q_j(x)\}_{j=1}^{\infty}$ of Faber–Schauder polynomials,

$$Q_j(x) = \sum_{n=m_{j-1}}^{m_j-1} A_n \varphi_n(x)$$

such that

$$|A_n| < 2^{-2j}, \ \forall n \in [m_{j-1}, m_j),$$

$$\left\| f(x) - \sum_{j=1}^{n} Q_j(x) \right\| \leq 2^{-(n+1)},$$

and, for each $B_n \subset \{m_{n-1}, \ldots, m_n - 1\}$,

$$\left\| \sum_{n \in B_n} A_n \varphi_n(x) \right\| < \frac{3}{2^n}.$$

As $n \to \infty$, $j \to \infty$ thus A_n converges to 0.

Further, from this it follows that the series

$$\sum_{n=m_0}^{\infty} A_n \varphi_n = \sum_{j=1}^{\infty} \sum_{n=m_{j-1}}^{m_j-1} A_n \varphi_n$$

converges unconditionally to $f(x)$ in the norm $L[0,1]$.

Indeed, if π is a permutation of $\mathbb{N}$, then we choose N_n so that $\{\pi(k) \colon m_0 \leq k < N_n\} \supset \{i \colon m_0 \leq i < m_n\}$. Thus, for arbitrary $M > N_n$ we obtain

$$\left\| f(x) - \sum_{k=m_0}^{M} A_{\pi(k)} \varphi_{\pi(k)}(x) \right\| \leq$$

$$\leq \left\| f(x) - \sum_{j=1}^{n} Q_j(x) \right\| + \sum_{j=n+1}^{\infty} \frac{3}{2^j} < \frac{1}{2^{n+1}} + \frac{3}{2^n} = \frac{7}{2^n}.$$

Letting $M \to \infty$, $n \to \infty$, thus we get that for every permutation $\pi(k)$ the series $\sum_{k=m_0}^{M} A_{\pi(k)} \varphi_{\pi(k)}(x)$ converges to $f(x)$ in L^1. □

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ՖԱԲԵՐ–ՇԱՈՒԴԵՐԻ ՇԱՐՔԵՐԻ ՈՉ ՊԱՅՄԱՆԱԿԱՆ ԶՈՒԳԱՄԻՏՈՒԹՅՈՒՆԸ L^1-ՈՒՄ

Այս աշխատանքում ապացուցված է, որ Ֆաբեր–Շաուդերի ֆունկցիաները կազմում են ոչ պայմանական ներկայացման համակարգ L^1-ի համար:

Т. М. ГРИГОРЯН, А. А. МАРАНДЖЯН

О БЕЗУСЛОВНОЙ СХОДИМОСТИ РЯДОВ ФАБЕРА–ШАУДЕРА В L^1

В работе доказано, что функции Фабера–Шаудера являются системой безусловных представлений для L^1.