PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2022, **56**(1), p. 19–26

Mathematics

ON INTERVAL EDGE-COLORINGS OF COMPLETE MULTIPARTITE GRAPHS

L. N. MURADYAN

Chair of Discrete Mathematics and Theoretical Informatics, YSU, Armenia

A graph G is called a complete r-partite ($r \geq 2$) graph, if its vertices can be divided into r non-empty independent sets $V_1, \ldots, V_r$ in a way that each vertex in V_i is adjacent to all the other vertices in V_j for $1 \leq i < j \leq r$. Let $K_{n_1,n_2,\ldots,n_r}$ denote a complete r-partite graph with independent sets $V_1, V_2, \ldots, V_r$ of sizes $n_1, n_2, \ldots, n_r$. An edge-coloring of a graph G with colors $1, 2, \ldots, t$ is called an *interval t-coloring*, if all colors are used and the colors of edges incident to each vertex of G are distinct and form an interval of integers.

In this paper we have obtained some results on the existence and construction of interval edge-colorings of complete r-partite graphs. Moreover, we have also derived an upper bound on the number of colors in interval colorings of complete multipartite graphs.

https://doi.org/10.46991/PYSU:A/2022.56.1.019

MSC2010: Primary 05C15.

Keywords: complete multipartite graph, edge-coloring, proper edge-coloring, interval coloring.

Introduction. A *proper edge-coloring* of a graph G is a mapping $\alpha : E(G) \to \mathbb{N}$, such that for every pair of adjacent edges $e, e' \in E(G)$, $\alpha(e) \neq \alpha(e')$. A proper edge-coloring of a graph G with colors $1, 2, \ldots, t$ is called an *interval t-coloring*, if all colors are used and the colors of edges incident to each vertex of G form an interval of integers. A graph G is *interval colorable*, if it has an interval t-coloring for some positive integer t. For interval colorable graphs, let $W(G)$ be the largest value of t for which G has an interval t-coloring.

In 1987 Asratian and Kamalian [1] introduced the concept of interval edge-coloring of graphs. In [2] Kamalian investigated interval edge-colorings of complete bipartite graphs and trees. Later, for general graphs, in [3] Kamalian obtained an upper bound on $W(G)$.

In [4] Petrosyan investigated interval edge-colorings of complete graphs and hypercubes. Later, in [5] authors improved the results for hypercubes. In [6]

* E-mail: levonmuradyanlevon@gmail.com

Khachatrian and Petrosyan improved lower and upper bounds for the number of colors of interval edge-colorings of complete graphs. In [7, 8] authors obtained some results on interval edge-colorings of complete 3-partite graphs. In [9] Petrosyan investigated interval edge-colorings of complete balanced multipartite graphs. In [10] Axenovich investigated interval edge-colorings of planar graphs. Recently, in [11] Sahakyan and Muradyan investigated interval edge-colorings of even block graphs.

There are also some results showing NP-completeness of problems on the existance of interval colorings. For example, in [12] Sevast'janov proved that it is a NP-complete problem to decide whether a bipartite graph has an interval coloring or not. Recently, in [13] Sahakyan and Muradyan proved that it is a NP-complete problem to decide whether complete or complete bipartite graphs have an interval coloring or not when there are restrictions on the edges, and the edge-coloring should satisfy those restrictions.

Notation, Definitions and Auxiliary Results. All graphs considered in this paper are undirected, finite, and have no loops or multiple edges. For an undirected graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. The maximum degree of G is denoted by $\Delta(G)$. Not defined terms and concepts can be found in [14].

A graph G is called a complete r-partite ($r \geq 2$) graph, if its vertices can be divided into r non-empty independent sets $V_1, \ldots, V_r$ such that each vertex in V_i is adjacent to all the other vertices in V_j for $1 \leq i < j \leq r$.

A *proper edge-coloring* of a graph G is a mapping $\alpha : E(G) \to \mathbb{N}$, such that for every pair of adjacent edges $e, e' \in E(G)$, $\alpha(e) \neq \alpha(e')$. A proper edge-coloring of a graph G with colors $1, 2, \ldots, t$ is called an *interval t-coloring*, if all colors are used and the colors of edges incident to each vertex of G form an interval of integers. A graph G is *interval colorable*, if it has an interval t-coloring for some positive integer t. Let $\mathfrak{N}$ is the set of all interval colorable graphs.

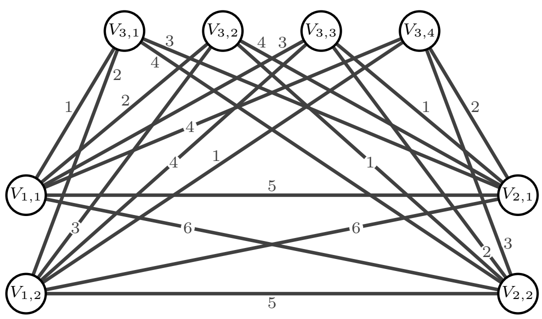

Fig. 1. The complete 3-partite graph $K_{2,2,4}$ with its interval coloring α and $LSE(V(K_{2,2,4}), \alpha) = (1, 1, 1, 1, 1, 1, 1, 1)$.

For a graph G, let $w(G)$ and $W(G)$ be the smallest and largest values of t, for which G has an interval t-coloring, respectively.

An ordered sequence of non negative integers $L=(l_1,l_2,\dots,l_k)$ is called a *continuous sequence*, if it contains all integers between the smallest and largest elements of L.

Let α be an interval edge-coloring of G and v be a vertex in G. Then the set of colors of edges incident to v is called the spectrum of a vertex v and denoted by $S(v,\alpha)$. The smallest and largest colors of $S(v,\alpha)$ are denoted by $\underline{S}(v,\alpha)$ and $\overline{S}(v,\alpha)$, respectively. Let $V'=\{v_1,\dots,v_k\}\in V(G)$. Let us define ordered sequence $LSE(V',\alpha)$ (Lower Spectral Edge) example shown in Fig. 1 in the following way:

$$LSE(V',\alpha)=(\underline{S}(v_{i_1},\alpha),\underline{S}(v_{i_2},\alpha),\dots,\underline{S}(v_{i_k},\alpha)),$$

where $\underline{S}(v_{i_l},\alpha)\le\underline{S}(v_{i_{l+1}},\alpha)$ for $1\le l<k$.

In [3] Kamalian proved the following for general graphs:

Theorem 1. *If G is a connected graph with at least two vertices and $G\in\mathfrak{N}$, then*

$$W(G)\le 2|V(G)|-3.$$

In this paper we improve this upper bound for complete multipartite graphs.

In 2012 Petrosyan [9] obtained the following result:

Theorem 2. *If $K_{n,\dots,n}$ is a complete balanced k-partite graph, then $K_{n,\dots,n}\in\mathfrak{N}$ if and only if nk is even. Moreover, if nk is even, then $w(K_{n,\dots,n})=n(k-1)$ and $W(K_{n,\dots,n})\ge\left(\frac{3}{2}k-1\right)n-1$.*

In [7] Grzesik and Khachatrian obtained the following result on interval colorings of complete 3-partite graphs:

Theorem 3. *For any $l,m\in\mathbb{N}$, $K_{l,m,l+m}\in\mathfrak{N}$.*

In [7] authors also posed the following conjecture:

Conjecture 1. *The graph $K_{l,m,n}$, where $l\le m\le n$ and $n>l+m$ is interval colorable if and only if the graph $K_{l,m,n-l-m}$ is interval colorable.*

In [15] Tepanyan and Petrosyan obtained the following lemma for complete bipartite graphs, which we will use later in the proof of our results.

Lemma 1. *If $K_{n,n}$ is a complete bipartite graph with bipartition (U,V), then for any continuous sequence L with length n, $K_{n,n}$ has an interval coloring α such that $LSE(U,\alpha)=LSE(V,\alpha)=L$.*

Main Results. Let us begin with an upper bound on $W(G)$ for complete multipartite graphs.

Theorem 4. *If $K_{n_1,n_2,\dots,n_r}$ is a complete r-partite graph with $n_1\ge n_2\ge\dots\ge n_r$ ($r\ge 2$) and $K_{n_1,n_2\dots,n_r}\in\mathfrak{N}$, then*

$$W(K_{n_1,n_2,\dots,n_r})\le 2\sum_{i=1}^{r}n_i-n_r-n_{r-1}-1.$$

Proof. Let denote $K_{n_1,n_2,\dots,n_r}$ by G for convenience.

Let α be an interval $W(G)$-coloring of G. Also let $v \in V_{i_0}$ (for some $1 \le i_0 \le r$) be a vertex such that $\underline{S}(v,\alpha) = 1$. By the definition of the spectrum of the vertex, we have

$$\overline{S}(v,\alpha) = \underline{S}(v,\alpha) + d_G(v) - 1 = 1 + \sum_{i=1}^{r} n_i - n_{i_0} - 1 = \sum_{i=1}^{r} n_i - n_{i_0}.$$

Let us consider an arbitrary vertex $u \in V_{j_0}$, where $1 \le j_0 \le r$ and $i_0 \ne j_0$. Since $(v,u) \in E(G)$, we can note that $\underline{S}(u,\alpha) \le \overline{S}(v,\alpha)$ and by that inequality, we can give the following upper bound for $\overline{S}(u,\alpha)$:

$$\overline{S}(u,\alpha) = \underline{S}(u,\alpha) + d_G(u) - 1 \le \overline{S}(v,\alpha) + d_G(u) - 1 \le$$
$$\le \sum_{i=1}^{r} n_i - n_{i_0} + \sum_{i=1}^{r} n_i - n_{j_0} - 1 \le 2\sum_{i=1}^{r} n_i - n_{i_0} - n_{j_0} - 1 \le$$
$$\le 2\sum_{i=1}^{r} n_i - n_r - n_{r-1} - 1.$$

Let us consider an arbitrary vertex $v' \in V_{i_0}$. By definition of the spectrum of the vertex, we get the following upper bound for $\overline{S}(v',\alpha)$:

$$\overline{S}(v',\alpha) \le \max_{u:(v',u)\in E(G)} \overline{S}(u,\alpha) = \max_{u \notin V_{i_0}} \overline{S}(u,\alpha) \le 2\sum_{i=1}^{r} n_i - n_r - n_{r-1} - 1.$$

Since for each vertex $v \in V(G)$, $\overline{S}(v,\alpha) \le 2\sum_{i=1}^{r} n_i - n_r - n_{r-1} - 1$, taking into account that $W(G) = \max_{v\in V(G)} \overline{S}(v,\alpha)$, we obtain

$$W(G) = \max_{v\in V(G)} \overline{S}(v,\alpha) \le 2\sum_{i=1}^{r} n_i - n_r - n_{r-1} - 1.$$

□

Let us note that Theorem 4 improves an upper bound in Theorem 1 for complete multipartite graphs (for example, when $n_{r-1} \ge 2$). Let us try to find a tight example for complete balanced r-partite graphs G, using Theorem 2:

$$\left(\frac{3}{2}r - 1\right)n - 1 \le W(G) \le 2nr - n - n - 1,$$
$$\left(\frac{3}{2}r - 1\right)n - 1 = 2nr - n - n - 1,$$
$$\frac{3}{2}nr - n - 1 = 2nr - 2n - 1,$$
$$n = \frac{nr}{2}.$$

From this equality, it follows that equation occurs when $r = 2$. We think that this upper bound become equality if and only if $r = 2$ and can be improved for the remaining cases.

Let us continue with the result about the existence and construction of interval colorings of complete multipartite graphs.

Theorem 5. *For any $n_1, n_2, \ldots, n_r \in \mathbb{N}$, if $K_{n_1,\ldots,n_r}$ has an interval t-coloring α such that $LSE(V(K_{n_1,\ldots,n_r}), \alpha)$ is continuous and $\sum_{i=1}^{r} n_i = n$, then for any $k \in \mathbb{Z}_{\geq 0}$, $K_{n_1,\ldots,n_r,n,2n,\ldots,2^k n}$ has an interval $(t+(2^{k+1}-1)n)$-coloring β such that $LSE(V(K_{n_1,\ldots,n_r,n,2n,\ldots,2^k n}), \beta)$ is continuous.*

Proof. We prove this Theorem by induction on k. First we show the statement of Theorem 5 for the case $k = 0$. Let us divide the vertices of $K_{n_1,\ldots,n_r,n}$ into the following two groups (Fig. 2):

- $U_1 = \bigcup_{i=1}^{r} V_i$, $|U_1| = \sum_{i=1}^{r} n_i = n$;
- $U_2 = V_{r+1}$, $|U_2| = |V_{r+1}| = n$.

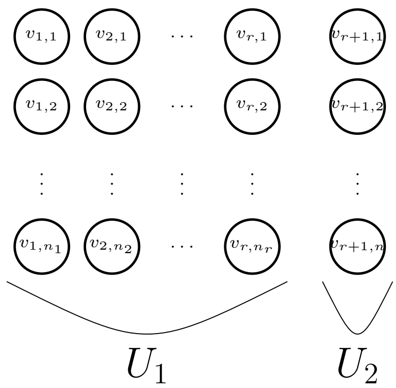

Fig. 2. Grouping of the vertices of $K_{n_1,n_2,\ldots,n_r,n}$.

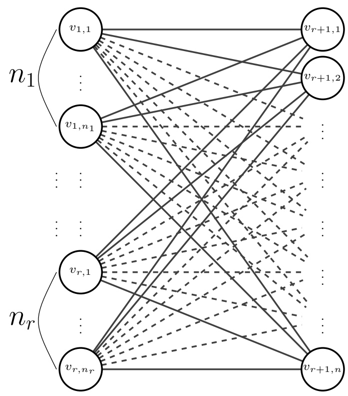

Fig. 3. Complete bipartite graph G with bipartition (U_1, U_2).

Define a complete bipartite graph G with bipartition (U_1, U_2) (Fig. 3). Clearly, G is isomorphic to $K_{n,n}$. By Lemma 1, we can give an interval coloring γ of G such that the following two conditions hold:

- $LSE(U_1, \gamma) = LSE(U_2, \gamma) = LSE(V(K_{n_1,\ldots,n_r}), \alpha)$;
- for any vertex $v \in U_1$, $\underline{S}(v, \gamma) = \underline{S}(v, \alpha)$.

We define an edge-coloring β of $K_{n_1,\ldots,n_r,n}$ as follows:

- for any vertices $v, u \in U_1$ such that $(v,u) \in E(K_{n_1,\ldots,n_r,n})$, let $\beta((v,u)) = \alpha((v,u)) + n$;
- for any vertex $v \in U_1$ and $u \in U_2$, let $\beta((v,u)) = \gamma((v,u))$.

Clearly, β is an interval $(t+n)$-coloring of $K_{n_1,\ldots,n_r,n}$. By the definition of β, it is easy to see that $LSE(V(K_{n_1,\ldots,n_r,n}), \beta)$ is continuous.

Assume that the statement of Theorem 5 holds for $k = l$. Let us consider the case $k = l+1$. By induction, we have that $K_{n_1,\ldots,n_r,n,2n,\ldots,2^l n}$ has an interval $(t+(2^{l+1}-1)n)$-coloring β such that $LSE(V(K_{n_1,\ldots,n_r,n,2n,\ldots,2^l n}), \beta)$ is continuous. Using the same argument as in the proof of the case $k = 0$, it can be easily obtained that the Theorem also holds in the case $k = l+1$. $\square$

Corollary 1. *For any* $n_1,\dots,n_r \in \mathbb{N}$, *if* $K_{n_1,\dots,n_r}$ *has an interval* $\Delta(K_{n_1,\dots,nd_r})$-*coloring* α *such that* $LSE(V(K_{n_1,\dots,n_r}),\alpha)$ *is continuous and* $\sum_{i=1}^{r} n_i = n$, *then for any* $k \in \mathbb{Z}_{\geq 0}$, $K_{n_1,\dots,n_r,n,2n,\dots,2^k n}$ *has an interval* $\Delta(K_{n_1,\dots,n_r,n,2n,\dots,2^k n})$-*coloring* β *such that* $LSE(V(K_{n_1,\dots,n_r,n,2n,\dots,2^k n}),\beta)$ *is continuous.*

Corollary 2. *For any* $a,b,c,k \in \mathbb{N}$, *if* $K_{a,b,c}$ *has an interval* t-*coloring* α *such that* $LSE(V_1 \cup V_2,\alpha)$ *is continuous, then* $K_{a,b,c+k(a+b)}$ *has an interval* $(t+a+b)$-*coloring* β *such that* $LSE(V_1 \cup V_2,\beta)$ *is continuous.*

We prove this Corollary by induction on k also using the proof of Theorem 5 in the case $k = 0$. Let us also note that this Corollary partially confirmes Conjecture 1.

We would like to thank Petros Petrosyan for his suggestions and support.

Received 02.12.2021
Reviewed 21.02.2022
Accepted 07.03.2022

REFERENCES

1. Asratian A.S., Kamalian R.R. Interval Colorings of Edges of a Multigraph. *Appl. Math.* **5** (1987), 25–34 (in Russian).
https://arxiv.org/abs/1401.8079
2. Kamalian R.R. *Interval Colorings of Complete Bipartite Graphs and Trees.* Preprint of the Computing Centre of the Academy of Sciences of Armenia. Yerevan (1989).
https://arxiv.org/abs/1308.2541
3. Kamalian R.R. *Interval Edge-colorings of Graphs*. PhD Thesis. Novosibirsk (1990).
4. Petrosyan P.A. Interval Edge-colorings of Complete Graphs and n-dimensional Cubes. *Discrete Math.* **310** (2010), 1580–1587.
https://doi.org/10.1016/j.disc.2010.02.001
5. Petrosyan P.A. , Khachatrian H.H. , Tananyan H.G. Interval Edge-colorings of Cartesian Products of Graphs I. *Discuss. Math. Graph Theory* **33** (2013), 613–632.
https://doi.org/10.7151/dmgt.1693
6. Khachatrian H.H., Petrosyan P.A. Interval Edge-colorings of Complete Graphs. *Discrete Mathematics* **339** (2016), 2249–2262.
https://doi.org/10.1016/j.disc.2016.04.002
7. Grzesik A., Khachatrian H. Interval Edge-colorings of $K_{1,m,n}$. *Discrete Appl. Math.* **174** (2014), 140–145.
https://doi.org/10.1016/j.dam.2014.04.003
8. Feng Y., Huang Q. Consecutive Edge-coloring of the Generalized θ-graphs. *Discrete Appl. Math.* **155** (2007), 2321–2327.
https://doi.org/10.1016/j.dam.2007.06.010
9. Petrosyan P.A. Interval Colorings of Complete Balanced Multipartite Graphs (2012).
https://arxiv.org/pdf/1211.5311.pdf

10. Axenovich M.A. On Interval Colorings of Planar Graphs. *Congr. Numer.* **159** (2002), 77–94.
https://www.math.kit.edu/iag6/~axenovich/media/interval_coloring.pdf
11. Sahakyan A.Kh., Muradyan L.N. Interval Edge-coloring of Even Block Graphs. *Norwegian Journal of Development of the International Science* **70** (2021), 26–30.
https://doi.org/10.24412/3453-9875-2021-70-1-26-30
12. Sevast'janov S.V. Interval Colorability of the Edges of a Bipartite Graph. *Metody Diskret. Analiza* **50** (1990), 61–72 (in Russian).
13. Sahakyan A.Kh., Muradyan L.N. Interval Edge-coloring of Complete and Complete Bipartite Graphs with Restrictions. *World Science* **9** (2021).
https://doi.org/10.31435/rsglobal_ws/30092021/7689
14. West D.B. *Introduction to Graph Theory*. NJ, Prentice-Hall (2001).
https://ia801204.us.archive.org/35/items/igt_west/igt_west_text.pdf
15. Tepanyan H.H. , Petrosyan P.A. Interval Edge-colorings of Composition of Graphs. *Discrete Appl. Math.* **127** (2017), 368–374.
https://doi.org/10.1016/j.dam.2016.09.022

Լ. Ն. ՄՈՒՐԱԴՅԱՆ

ԼՐԻՎ ԲԱԶՄԱԿՈՂՄԱՆԻ ԳՐԱՖՆԵՐԻ ՄԻՋԱԿԱՅՔԱՅԻՆ ԿՈՂԱՅԻՆ ՆԵՐԿՈՒՄՆԵՐԻ ՄԱՍԻՆ

G գրաֆը կոչվում է *լրիվ r-կողմանի* ($r \geq 2$) գրաֆ, եթե գրաֆի գագաթների բազմությունը կարելի է տրոհել r ոչ դատարկ անկախ բազմությունների $V_1, \ldots, V_r$ այնպես, որ կամայական գագաթ V_i-ից հարևան է բոլոր գագաթներին V_j-ից $(1 \leq i < j \leq r)$: Լրիվ r-կողմանի գրաֆը $V_1, V_2, \ldots, V_r$ անկախ բազմություններով, որտեղ $|V_i| = n_i$ $(1 \leq i \leq r)$, կնշանակենք $K_{n_1,n_2,\ldots,n_r}$-ով: G գրաֆի կողային ներկումը $1, 2, \ldots, t$ գույներով կոչվում է *միջակայքային t-ներկում է*, եթե բոլոր գույները օգտագործված են, և G-ի կամայական գագաթին կից կողերի գույները տարբեր են և կազմում են ամբողջ թվերի միջակայք:

Այս աշխատանքում ստացվել են լրիվ r-կողմանի գրաֆների միջակայքային կողային ներկումների գոյության, կառուցման և թվային պարամետրերի գնահատման հետ կապված որոշ արդյունքներ: Մասնավորապես, աշխատանքում ստացվել է լրիվ բազմակողմանի գրաֆների միջակայքային կողային ներկման մեջ մասնակցող հնարավոր գույների քանակի վերին գնահատական:

Л. Н. МУРАДЯН

ОБ ИНТЕРВАЛЬНЫХ РЕБЕРНЫХ РАСКРАСКАХ ПОЛНЫХ МНОГОДОЛЬНЫХ ГРАФОВ

Граф G называется *полным r-дольным ($r \geq 2$) графом*, если множество его вершины можно разбить на r непустых независимых множеств $V_1, \ldots, V_r$ таким образом, что каждая вершина из V_i смежна со всеми вершинами из V_j $(1 \leq i < j \leq r)$. Полный r-дольный граф с независимыми множествами $V_1, V_2, \ldots, V_r$, где $|V_i| = n_i$ $(1 \leq i \leq r)$, обозначим через $K_{n_1,n_2,\ldots,n_r}$. Реберная раскраска графа G в цвета $1, 2, \ldots, t$ называется *интервальной t-раскраской*, если все цвета использованы и цвета ребер, инцидентных любой вершине графа G, различны и образуют интервал целых чисел.

В настоящей статье получены некоторые результаты, касающиеся задач существования, построения и оценки числовых параметров интервальных реберных раскрасок полных r-дольных графов. Кроме того, нами также получена верхняя оценка числа цветов в интервальных реберных раскрасках полных многодольных графов.