PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2022, **56**(3), p. 97–106

Mathematics

ON A RESULT CONCERNING ALGEBRAIC CURVES PASSING THROUGH n-INDEPENDENT NODES

H. A. HAKOPIAN

Chair of Numerical Analysis and Mathematical Modelling, YSU, Armenia

Let a set of nodes $\mathcal{X}$ in the plane be n-independent, i.e. each node has a fundamental polynomial of degree n. Assume that $\#\mathcal{X} = d(n,n-3)+3 = (n+1)+n+\cdots+5+3$. In this paper we prove that there are at most three linearly independent curves of degree less than or equal to $n-1$ that pass through all the nodes of $\mathcal{X}$. We provide a characterization of the case when there are exactly three such curves. Namely, we prove that then the set $\mathcal{X}$ has a very special construction: either all its nodes belong to a curve of degree $n-2$, or all its nodes but three belong to a (maximal) curve of degree $n-3$.

This result complements a result established recently by H. Kloyan, D. Voskanyan, and H. Hakopian. Note that the proofs of the two results are completely different.

https://doi.org/10.46991/PYSU:A/2022.56.3.097

MSC2010: 41A05, 41A63, 14H50.

Keywords: algebraic curve, maximal curve, fundamental polynomial, n-independent nodes.

Introduction. Denote the space of all bivariate polynomials of total degree $\leq n$ by

$$\Pi_n = \left\{ \sum_{i+j\leq n} a_{ij}x^iy^j \right\}.$$

We have that $N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2)$.

Denote by Π the space of all bivariate polynomials.

A *plane algebraic curve* is the zero set of some bivariate polynomial of degree ≥ 1. To simplify notation, we shall use the same letter, say p, to denote the polynomial $p \in \Pi$ and the curve given by the equation $p(x,y) = 0$. In particular, by ℓ we denote a linear polynomial from Π_1 and the line defined by the equation $\ell(x,y) = 0$.

* E-mail: hakop@ysu.am

Consider a set of s distinct nodes $\mathcal{X} = \mathcal{X}_s = \{(x_1,y_1),(x_2,y_2),\dots,(x_s,y_s)\}$. The problem of finding a polynomial $p \in \Pi_n$, which satisfies the conditions

$$p(x_i,y_i) = c_i, \quad i = 1,\dots,s, \tag{1}$$

is called interpolation problem.

Denote by $p\big|_{\mathcal{X}}$ the restriction of $p \in \Pi$ on $\mathcal{X}$.

A polynomial $p \in \Pi_n$ is called a fundamental polynomial for a node $A \in \mathcal{X}$ if $p(A) = 1$ and $p\big|_{\mathcal{X}\setminus\{A\}} = 0$.

We denote this n-fundamental polynomial by $p_A^\star := p_{A,\mathcal{X}}^\star$.

Definition 1. *The interpolation problem with a set of nodes $\mathcal{X}_s$ is called n-poised, if for any data $(c_1,\dots,c_s)$ there is a unique polynomial $p \in \Pi_n$ satisfying the interpolation Conditions* (1).

A necessary condition of poisedness is $\#\mathcal{X}_s = s = N$.

Now, let us consider the concept of n-independence (see [1, 2]).

Definition 2. *A set of nodes $\mathcal{X}_s$ is called n-independent, if all its nodes have n-fundamental polynomials. Otherwise, it is called n-dependent.*

Fundamental polynomials are linearly independent. Therefore, a necessary condition of n-independence for $\mathcal{X}_s$ is $s \le N$.

In this paper we consider n-independence more generally. Namely, we admit possibility to include in the n-independent set $\mathcal{X}_s$ a *directional derivative node*, denoted by $A^{(k)}$. We have that $p(A^{(k)}) := D_{\mathbf{a}}^k p(A)$, where $p \in \Pi$, $\mathbf{a}$ is a direction, and $k \in \mathbb{N}$. For a node $A^{(k)}$ we assume in addition that

$$p \in \Pi_n,\ p|_{\mathcal{X}} = 0 \implies D_{\mathbf{a}}^i p(A) = 0,\ i = 0,\dots,k-1.$$

The set $\mathcal{X} \cup \{A^{(k)}\}$ is n-independent means that $\mathcal{X}$ is n-independent and the node $A^{(k)}$ has an n-fundamental polynomial $p = p_{A^{(k)}}^\star$:

$$p \in \Pi_n,\ p|_{\mathcal{X}} = 0,\ D_{\mathbf{a}}^k p(A) = 1.$$

We say that a node $A^{(k)}$ belongs to a curve q if $D_{\mathbf{a}}^i p(A) = 0,\ i = 0,\dots,k$. In particular $A^{(k)}$ belongs to a line ℓ, if $A \in \ell$ and $\mathbf{a}$ is the direction vector of ℓ.

Let us mention, as it can be readily verified, that all the results we present below concerning n-independent sets hold true for the above mentioned generalization.

Some Properties of n-Independent Nodes. Let us start with the following

Lemma 1. (Lemma 2.2, [3]). *Suppose that a set of nodes $\mathcal{X}$ is n-independent and a node $A \notin \mathcal{X}$ has an n-fundamental polynomial with respect to the set $\mathcal{X} \cup \{A\}$. Then the latter set is n-independent too.*

Denote the distance between the points A and B by $\rho(A,B)$. Let us recall the following (see Rem. 1.14, [4]).

Lemma 2. *Suppose that $\mathcal{X}_s = \{A_i\}_{i=1}^s$ is an n-independent set. Then there is a number $\varepsilon > 0$ such that any set $\mathcal{X}'_s = \{A'_i\}_{i=1}^s$, with the property that $\rho(A_i,A'_i) < \varepsilon,\ i = 1,\dots,s$, is n-independent too.*

Next result concerns the extensions of n-independent sets.

Lemma 3. (Lemma 2.1, [2]). *Any n-independent set $\mathcal{X}$ with $\#\mathcal{X} < N$ can be enlarged to an n-poised set.*

Denote the linear space of polynomials of total degree at most n vanishing on $\mathcal{X}$ by

$$\mathcal{P}_{n,\mathcal{X}} = \left\{ p \in \Pi_n : p\big|_{\mathcal{X}} = 0 \right\}.$$

The following two propositions are well-known (see, e.g., [2]).

Proposition 1. *For any node set $\mathcal{X}$ we have that*

$$\dim \mathcal{P}_{n,\mathcal{X}} = N - \#\mathcal{Y},$$

where $\mathcal{Y}$ is a maximal n-independent subset of $\mathcal{X}$.

Proposition 2. *If a polynomial $p \in \Pi_n$ vanishes at $n+1$ points of a line ℓ, then we have that $p = \ell r$, where $r \in \Pi_{n-1}$.*

In the sequel we will need the following

Proposition 3. (Prop. 1.10, [3]). *Let $\mathcal{X}$ be a set of nodes. Then the following two conditions are equivalent:*
$i)$ $\mathcal{P}_{n,\mathcal{X}} = \{0\}$;
$ii)$ the node set $\mathcal{X}$ has an n-poised subset.

Set $d(n,k) := N_n - N_{n-k} = (1/2)k(2n+3-k)$. The following is a generalization of Proposition 2.

Proposition 4. (Prop. 3.1, [5]). *Let q be an algebraic curve of degree $k \le n$ without multiple components. Then the following hold:*
$i)$ any subset of q containing more than $d(n,k)$ nodes is n-dependent;
$ii)$ any subset $\mathcal{X}$ of q containing exactly $d = d(n,k)$ nodes is n-independent if and only if the following condition holds:

$$p \in \Pi_n \quad \text{and} \quad p|_{\mathcal{X}} = 0 \Longrightarrow p = qr, \text{ where } r \in \Pi_{n-k}. \tag{2}$$

Thus, according to Proposition 4, $i)$, at most $d(n,k)$ nodes of $\mathcal{X}$ can lie in a curve q of degree $k \le n$. This motivates the following

Definition 3. (Def. 3.1, [5]). *Given an n-independent set of nodes $\mathcal{X}_s$ with $s \ge d(n,k)$. A curve of degree $k \le n$ passing through $d(n,k)$ points of $\mathcal{X}_s$ is called maximal.*

We say that a node A of an n-poised set $\mathcal{X}$ uses a curve $q \in \Pi_k$, if the latter divides the n-fundamental polynomial of A, i.e. $p_A^\star = qr,\ r \in \Pi_{n-k}$.
Let us bring a characterization of maximal curves:

Proposition 5. (Prop. 3.3, [5]). *Let a node set $\mathcal{X}$ be n-independent. Then a curve μ of degree k, $k \le n$, is a maximal curve if and only if*

$$p \in \Pi_n,\ p|_{\mathcal{X}\cap\mu} = 0 \implies p = \mu q,\ q \in \Pi_{n-k}.$$

Next result concerns maximal independent sets in curves.

Proposition 6. (Prop. 3.5, [6]). *Assume that σ is an algebraic curve of degree k without multiple components and $\mathcal{X}_s \subset \sigma$ is any n-independent node set of cardinality $s,\ s < d(n,k)$. Then the set $\mathcal{X}_s$ can be extended to a maximal n-independent set $\mathcal{X}_d \subset \sigma$ of cardinality $d = d(n,k)$.*

Next result from Algebraic Geometry will be used in the sequel:

Theorem 1. (Th. 2.2, [7]). *If $\mathcal{C}$ is a curve of degree n with no multiple components, then through any point O not in $\mathcal{C}$ there pass lines which intersect $\mathcal{C}$ in n distinct points.*

Let us mention that, as it follows from the proof, if a line ℓ through the point O intersects $\mathcal{C}$ in n distinct points then any line through O, sufficiently close to ℓ, has the same property.

Finally, let us present a well-known

Lemma 4. *Given m linearly independent polynomials, $m \geq 2$. Then for any point A there are $m-1$ linearly independent polynomials, in their linear span, vanishing at A.*

A Result and Its Complement. In this paper we complement the following

Theorem 2. (Theorem 2.5, [8]). *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-2)+3$ nodes with $3 \leq k \leq n-2$. Then at most three linearly independent curves of degree $\leq k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such three curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ lie in a curve of degree $k-1$, or all the nodes of $\mathcal{X}$ but three lie in a (maximal) curve of degree $k-2$.*

Namely, we prove that the above result is true also in the case $k = n-1$:

Proposition 7. *Assume that $\mathcal{X}$ is an n-independent set of $d(n,n-3)+3$ nodes, $n \geq 4$. Then at most three linearly independent curves of degree $\leq n-1$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such three curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ lie in a curve of degree $n-2$, or all the nodes of $\mathcal{X}$ but three lie in a (maximal) curve of degree $n-3$.*

In the sequel we will use the following

Theorem 3. (Theorem 3, [9]). *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-2)+2$ nodes with $3 \leq k \leq n-1$. Then at most four linearly independent curves of degree $\leq k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such four curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but two lie in a maximal curve of degree $k-2$.*

Proof of Proposition 7. Assume to the contrary that there are four linearly independent curves of degree $\leq n-1$ passing through all the nodes of the n-independent set $\mathcal{X}$ with $\#\mathcal{X} = d(n,n-3)+3$. Then, according to Theorem 3, all the nodes of $\mathcal{X}$ but three belong to a maximal curve μ of degree $n-3$. The curve μ is maximal and the remaining three nodes of $\mathcal{X}$, denoted by A,B and C, are outside of it: $A,B,C \notin \mu$. Hence we have that

$$\mathcal{P}_{n-1,\mathcal{X}} = \{p \in \Pi_{n-1} : p_{\mathcal{X}} = 0\} = \{q\mu : q \in \Pi_2,\ q(A) = q(B) = q(C) = 0\}.$$

Thus, we readily get that $\dim \mathcal{P}_{n-1,\mathcal{X}} = \dim\{q \in \Pi_2 : q(A) = q(B) = q(C) = 0\}$ $= \dim \mathcal{P}_{2,\{A,B,C\}} = 6-3 = 3$, which contradicts our assumption. Note that in the last equality we use Proposition 1 and the fact that any three nodes are 2-independent.

Now, let us verify the part "if". By assuming that there is a curve σ of degree $n-3$ passing through the nodes of $\mathcal{X}$, we readily find three linearly independent curves of degree $\leq n-1$: $\sigma, x\sigma, y\sigma$, passing through $\mathcal{X}$. While if we assume that all the nodes of $\mathcal{X}$ but three lie in a curve μ of degree $n-3$, then above evaluation shows that $\dim \mathcal{P}_{n-1,\mathcal{X}} = 3$. Note that till here the proof was similar to the proof of Theorem 2 in [8].

Finally, let us verify the part "only if". Denote the three curves passing through all the nodes of the set $\mathcal{X}$ by $\sigma_1, \sigma_2, \sigma_3$. If one of them is of degree $n-2$, then the conclusion of Theorem is satisfied and we are done. Thus, we may assume that each curve is of exact degree $n-1$ and has no multiple components.

We start with two nodes $B_1, B_2 \notin \mathcal{X}$, for which the following conditions are satisfied, where the line between B_1 and B_2 is denoted by ℓ_{12} :

i) the nodes B_1, B_2 do not belong to the curves $\sigma_1, \sigma_2, \sigma_3$;

ii) the set $\mathcal{X} \cup \{B_1, B_2\}$ is n-independent;

iii) the line ℓ_{12} does not pass through any node from $\mathcal{X}$;

iv) the line ℓ_{12} intersects each of the curves $\sigma_1, \sigma_2, \sigma_3$ at $n-1$ different points. Moreover, it intersects any two different components of these curves at different points.

Let us verify that one can find such two nodes. Indeed, in view of Lemma 3, we can start by choosing some nodes B_i', $i = 1,2$, satisfying the conditions *i*) and *ii*). Then, according to Lemma 2, for some positive ε any two nodes in the ε neighborhoods of B_i', $i = 1,2$, respectively, satisfy the first two conditions.

Next, from these neighborhoods, in view of Theorem 1, we can choose the nodes B_i, $i = 1,2$, satisfying the condition *iii*) and *iv*) too. Let us mention that to get the part "Moreover" of *iv*) we apply Theorem 1 for the curve consisting of all different components of the curves $\sigma_1, \sigma_2, \sigma_3$.

In the proof of Proposition we will later need the following

Lemma 5. *Assume that the hypotheses of Proposition 7 hold and assume additionally that at least one of the following conditions hold:*

(*a*) *a nontrivial linear combination of two polynomials from* $\{\sigma_1, \sigma_2, \sigma_3\}$, *denoted by* s_2, *vanishes at* B_1 *and* B_2 : $s_2(B_1) = s_2(B_2) = 0$;

(*b*) *a nontrivial linear combination of the polynomials* $\{\sigma_1, \sigma_2, \sigma_3\}$, *denoted by* s_3, *vanishes at* B_1, B_2, *and* $B_3 \in \ell_{12}$: $s_3(B_1) = s_3(B_2) = s_3(B_3) = 0$, *and the set* $\mathcal{X}''' := \mathcal{X} \cup \{B_1, B_2, B_3\}$ *is n-independent.*

Then we have that the statement of Proposition 7 holds.

Proof. Let us start with (b). In view of Proposition 6, we can extend the set $\mathcal{X}'''$ till a maximal n-independent set $\mathcal{Y} \subset s_3$ by adding $d(n,n-1) - (d(n,n-3) + 3) - 3 = 1$ node, denoted by B, i.e. $\mathcal{Y} = \mathcal{X}''' \cup \{B\}$.

Thus, s_3 is a maximal curve of degree $n-1$ for the node set $\mathcal{Y}$.

Then, in view of Lemma 4, we can find a nontrivial linear combination s of $\sigma_1, \sigma_2, \sigma_3$ such that s differs from s_3 and vanishes on $\mathcal{X} \cup \{B\}$.

Now consider the polynomial $s\ell_{12} \in \Pi_n$, which vanishes on the node set $\mathcal{Y}$. By Proposition 5, we conclude that

$$s\ell_{12} = s_3\ell, \text{ where } \ell \in \Pi_1.$$

The line ℓ_{12} differs from ℓ, since s differs from s_3. Therefore, we get that

$$s_3 = \ell_{12}q, \text{ where } q \in \Pi_{n-2}. \tag{3}$$

Now, by using $iii)$, we obtain that $q|_{\mathcal{X}} = 0$. Hence the statement of Proposition 7 holds.

(a) Assume, without loss of generality, that $s_2 := c_1\sigma_1 + c_2\sigma_2,\ s_2 \neq 0$, and $s_2(B_1) = s_2(B_2) = 0$.

Let us show that there is a node $B_3 \in \ell_{12}$ such that $s_2(B_3) \neq 0$. Indeed, assume conversely that $s_2|_{\ell_{12}} = 0$. Then, by Proposition 2, we obtain that

$$s_2 = \ell_{12}q,\ q \in \Pi_{n-2},$$

which finishes the proof in the same way as the relation (3).

Now, note that s_2 is a fundamental polynomial for $B_3 \in \mathcal{X}''' := \mathcal{X} \cup \{B_1, B_2, B_3\}$. By Lemma 1, the set $\mathcal{X}'''$ is n-independent.

Then assume, in view of Lemma 4, that s is a nontrivial linear combination of s_2 and σ_3 such that $s(B_3) = 0$, implying that $s|_{\mathcal{X}'''} = 0$. Thus the hypothesis of (b) is satisfied. □

Next, let us continue the proof of Proposition 7.

By using Lemma 4, consider a nontrivial linear combination of $\sigma_1, \sigma_2, \sigma_3$, denoted by s, that vanishes at B_1 and B_2. Set $\mathcal{X}'' := \mathcal{X} \cup \{B_1, B_2\}$.

Denote the set of intersection points of the line ℓ_{12} and the curve $s, \deg s = n-1$, by $\mathcal{I} := \ell_{12} \cap s$. We have that $\#\mathcal{I} = n-1$, counting also the multiplicities. Of course $B_1, B_2 \in \mathcal{I}$.

Case 1. First consider the case when one of B_1, B_2, say B_1, is a multiple point of intersection, i.e. $D_{\mathbf{a}}s(B_1) = 0$, where $\mathbf{a}$ is the direction vector of the line ℓ_{12}.

Let us prove that the set $\mathcal{Y} := \mathcal{X}'' \cup \{B_1^{(1)}\} = \mathcal{X} \cup \{B_1, B_2, B_1^{(1)}\}$ is n-independent, where $B_1^{(1)}$ means the directional derivative node with the direction $\mathbf{a}$ at B_1. According to Lemma 1 we need to point out a fundamental polynomial $q \in \Pi_n$, for $B_1^{(1)} \in \mathcal{Y}$, i.e. $q|_{\mathcal{X}''} = 0$ and $D_{\mathbf{a}}q(B_1) \neq 0$.

For this end consider a nontrivial polynomial $s_0 := c_1\sigma_1 + c_2\sigma_2,\ s_0 \neq 0$, which vanishes at $B_2:\ s_0(B_2) = 0$.

In view of Lemma 5 we may assume that $s_0(B_1) \neq 0$.

Then consider a line ℓ passing through B_1 with a direction vector different from $\mathbf{a}$. One can verify readily that the polynomial $q := \ell s_0$ is a desired polynomial. Indeed, we have that $q|_{\mathcal{X}''} = 0$. Then we have that

$$D_{\mathbf{a}} q(B_1) = D_{\mathbf{a}}[\ell s_0](B_1)$$
$$= (D_{\mathbf{a}}\ell)(B_1)s_0(B_1) + \ell(B_1)D_{\mathbf{a}}s_0(B_1) = (D_{\mathbf{a}}\ell)(B_1)s_0(B_1) \neq 0.$$

Thus the set $\mathcal{Y}$ is n-independent and hence this case can be proved in the same way as Lemma 5 (b).

Case 2. It remains to consider the case when both B_1 and B_2 are simple points of intersection. We have that $\#\mathcal{I} = n-1 \geq 3$. Consider another point of intersection of ℓ_{12} and $s:\ B \in \mathcal{I},\ B \neq B_1, B_2$.

In view of Lemma 5, (b), we may assume the following

Assumption 1. *The set $\mathcal{X}'' \cup \{B\} = \mathcal{X} \cup \{B_1, B_2, B\}$ is n-dependent.*

This here means that $\ p \in \Pi_n,\ p|_{\mathcal{X}''} = 0 \implies p(B) = 0.$

Now consider two nontrivial linear combinations s_1, s_2 of σ_1, σ_2 such that $s_1(B_2) = s_2(B_1) = 0.$

By Lemma 5, (a), we get that $s_i(B_i) \neq 0,\ i = 1,2$. Assume, without loss of generality, that $s_i(B_i) = 1,\ i = 1,2.$

Next let us show that $s_i(B) = 0,\ i = 1,2$. Let say $i = 1$. Consider the polynomial $q := \ell s_1 \in \Pi_n$, where the line ℓ passes through B_1 and does not pass through B. We have that $q(B_1) = q(B_2) = 0$. By using Assumption 1 and Lemma 1, we get $q(B) = 0$, hence $s_1(B) = 0$.

Now we are in a position to show that $\sigma_1(B) = \sigma_2(B) = \sigma_3(B) = 0$.

Let us show for example that $\sigma_1(B) = 0$.

Consider the polynomial $p = \sigma_1 - c_1 s_1 - c_2 s_2$, where $c_i = \sigma_1(B_i)$. We readily get that $p(B_1) = p(B_2) = 0$. Hence, in view of Assumption 1, as above, we get that $p(B) = 0$. It remains to note that $\sigma_1(B) = p(B) = 0$.

Next suppose that the point B is multiple:

$$s(B) = D_{\mathbf{a}} s(B) = \ldots, D_{\mathbf{a}}^{(k)} s(B) = 0,\ k \in \mathbb{N}.$$

In view of Lemma 5, (b), we may assume the following

Assumption 2. *The set $\mathcal{X}'' \cup \{B^{(i)}\},\ i = 0, \ldots, k$, is n-dependent.*

This here means that

$$p \in \Pi_n,\ p|_{\mathcal{X}''} = 0 \implies p(B) = D_{\mathbf{a}} p(B) = \ldots = D_{\mathbf{a}}^{(k)} p(B) = 0. \tag{4}$$

Now consider the above defined polynomials s_1 and s_2 with

$$s_1(B_1) = 1,\ s_1(B_2) = s_1(B) = 0, \quad s_2(B_2) = 1,\ s_2(B_1) = s_1(B) = 0.$$

By using induction on k, let us show that

$$D_{\mathbf{a}}^{(i)} s_j(B) = 0,\ i = 0, 1, \ldots, k,\ j = 1, 2. \tag{5}$$

Let say $j = 1$. The first step of induction is the above considered case $k = 0$. Assume that the case of $k-1$ is true, i.e. the first k equalities in (5) hold. Let us prove the last one, i.e. $D_{\mathbf{a}}^{(k)} p(B) = 0$.

Consider the polynomial $q := s_1\ell \in \Pi_n$, where the line ℓ passes through B_1 and does not pass through B. We have that $q(B_1) = q(B_2) = 0$. In view of Assumption 2, we get that

$$0 = D_{\mathbf{a}}^{(k)}[s_1\ell](B) = D_{\mathbf{a}}^{(k)}s_1(B)\ell(B) + kD_{\mathbf{a}}^{(k-1)}s_1(B)D_{\mathbf{a}}\ell(B) = D_{\mathbf{a}}^{(k)}s_1(B)\ell(B).$$

Since $\ell(B) \neq 0$ we conclude that $D_{\mathbf{a}}^{(k)}s_1(B) = 0$.

Now we are in a position to show that

$$D_{\mathbf{a}}^{(i)}\sigma_1(B) = D_{\mathbf{a}}^{(i)}\sigma_2(B) = D_{\mathbf{a}}^{(i)}\sigma_3(B) = 0,\ i = 0,1,\ldots,k. \tag{6}$$

Let us prove say equalities with σ_1. Consider the polynomial

$$p = \sigma_1 - c_1s_1 - c_2s_2, \text{ where } c_i = \sigma_1(B_i). \tag{7}$$

We readily get that $p(B_1) = p(B_2) = 0$. Hence, in view of Assumption 2, as above, we get $p(B) = D_{\mathbf{a}}p(B) = \ldots = D_{\mathbf{a}}^{(k)}p(B) = 0$. It remains to use the relations (5) and (7).

Hence except the two intersection points $B_1, B_2 \in \mathcal{I} := \ell_{12}\cap, s$ all other $n-3$ points, counting also the multiplicities, are common for the three curves σ_1, σ_2, and σ_3.

From this, in view of the Condition (iv), we conclude that the above three polynomials σ_1, σ_2, and σ_3 have a common divisor $q \in \Pi_{n-3}$:

$$\sigma_1 = \beta_1 q, \quad \sigma_2 = \beta_2 q, \quad \sigma_3 = \beta_3 q, \text{ where } \beta_i \in \Pi_2.$$

Therefore, we have

$$\mathcal{X} \subset \sigma_1 \cap \sigma_2 \cap \sigma_3 \subset q \cup [\beta_1 \cap \beta_2 \cap \beta_3]. \tag{8}$$

Now consider two cases for $\mathcal{B} := \beta_1 \cap \beta_2 \cap \beta_3$.

Case (a), $\#\mathcal{B} \geq 4$. According to Proposition 1 any subset $\mathcal{A} \subset \mathcal{B}$ with $\#\mathcal{A} = 4$ is 2-dependent. From here we readily obtain that the points of $\mathcal{A}$ are collinear. Hence all the points of $\mathcal{B}$ are collinear: $\mathcal{B} \subset \ell \in \Pi_1$.

Now we readily get that ℓ is a common divisor of β_1, β_2, and β_3, i.e.

$$\beta_1 = \ell_1\ell, \quad \beta_2 = \ell_2\ell, \quad \beta_3 = \ell_3\ell,$$

where $\ell_i \in \Pi_1$. Thus, as above, we get

$$\mathcal{B} \subset \ell \cup [\ell_1 \cap \ell_2 \cap \ell_3] \subset \ell. \tag{9}$$

The last relation here we get from the fact that the polynomials $\sigma_1, \sigma_2, \sigma_3$, and hence the polynomials $\ell_1, \ell_2\,\ell_3$ are linearly independent and hence $\ell_1 \cap \ell_2 \cap \ell_3 = \emptyset$.

Finally, we get from Eqs. (8) and (9) that

$$\mathcal{X} \subset q \cup \ell,$$

or, in other words, all the nodes of $\mathcal{X}$ lie in a curve of degree $n-2$, namely in the curve $q\ell \in \Pi_{n-2}$.

Case (b), $\#\mathcal{B} \leq 3$. In this case we obtain from Eq. (8) that all the nodes of $\mathcal{X}$ but ≤ 3 lie in a curve q of degree $n-3$. From here we readily conclude that q is a maximal curve and exactly 3 nodes of $\mathcal{X}$ are outside of it.

Thus Proposition 7 is proved. □

Finally note that in view of Theorem 2 and Proposition 7 one can formulate the following

Theorem 4. *Assume that $\mathcal{X}$ is an n-independent set of $d(n,k-2)+3$ nodes with $3 \le k \le n-1$. Then at most three linearly independent curves of degree $\le k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such three curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ lie in a curve of degree $k-1$, or all the nodes of $\mathcal{X}$ but three lie in a (maximal) curve of degree $k-2$.*

The work was supported by the Science Committee of RA, in the frames of the research project No. 21T-A055.

Received 22.09.2022
Reviewed 14.10.2022
Accepted 28.10.2022

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Հ. Ա. ՀԱԿՈԲՅԱՆ

n-ԱՆԿԱԽ ՀԱՆԳՈՒՅՑՆԵՐՈՎ ԱՆՑՆՈՂ ՀԱՆՐԱՀԱՇՎԱԿԱՆ ԿՈՐԵՐԻ ՄԱՍԻՆ ՄԻ ԱՐԴՅՈՒՆՔԻ ՎԵՐԱԲԵՐՅԱԼ

Դիցուք հանգույցների $\mathcal{X}$ բազմությունը հարթության վրա n-անկախ է, այսինքն յուրաքնչյուր հանգույց ունի n աստիճանի ֆունդամենտալ բազմանդամ: Ենթադրենք $\#\mathcal{X} = d(n,n-3)+3 = (n+1)+n+\cdots+5+3$: Այս հոդվածում ապացուցվում է, որ կան ամենաշատը երեք գծորեն անկախ $\leq n-3$ աստիճանի կորեր, որոնք անցնում են $\mathcal{X}$ բազմության բոլոր կետերով: Մենք բնութագրում ենք այն դեպքը երբ այդպիսի կորերը ճիշտ երեքն են: Այն է՝ ապացուցվում է, որ այդ դեպքում $\mathcal{X}$ բազմությունը ունի յուրահատուկ կառուցվածք. կամ իր բոլոր հանգույցները գտնվում են $n-2$ աստիճանի կորի վրա, կամ իր բոլոր հանգույցները, բացի երեքից, գտնվում են $n-3$ աստիճանի (մաքսիմալ) կորի վրա:

Այս արդյունքը լրացնում է Հ. Հակոբյանի, Հ. Քլոյանի և Դ. Ոսկանյանի կողմից վերջերս ապացուցված մի արդյունք: Նշենք, որ այս երկու արդյունքների ապացույցները ամբողջովին տարբեր են:

А. А. АКОПЯН

ОБ ОДНОМ РЕЗУЛЬТАТЕ ОТНОСИТЕЛЬНО АЛГЕБРАИЧЕСКИХ КРИВЫХ, ПРОХОДЯЩИХ ЧЕРЕЗ n-НЕЗАВИСИМЫЕ УЗЛЫ

Пусть множество узлов $\mathcal{X}$ на плоскости n-независимо, то есть каждый узел имеет фундаментальный многочлен степени n. Предположим, что $\#\mathcal{X} = d(n,n-3)+3 = (n+1)+n+\cdots+5+3$. В статье мы доказываем, что существуют не более трех линейно независимых кривых степени $\leq n-1$, которые проходят через все точки $\mathcal{X}$. Мы характеризуем случай, когда таких кривых ровно три. А именно, доказываем, что тогда множество $\mathcal{X}$ имеет особую конструкцию: либо все его точки лежат на кривой степени $n-2$, либо все его точки, кроме трех, лежат на (максимальной) кривой степени $n-3$.

Результат настоящей статьи дополняет результат, недавно полученный А. Акопяном, А. Клояном и Д. Восканяном. Отметим, что доказательства этих двух результатов совершенно различны.