

THE MOORE–PENROSE INVERSE OF TRIDIAGONAL  
SKEW-SYMMETRIC MATRICES. I

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The present work is devoted to deriving closed form expressions for the elements of the Moore–Penrose inverse of tridiagonal real skew-symmetric matrices. In the first part of the work we obtain results, concerning matrices of even order. A calculation approach for the generalized inverses of odd order matrices is provided.

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**Introduction.** A real matrix  $A$  is called *skew-symmetric* if  $A^T = -A$ . Obviously such matrices have zero diagonal elements.

Interest in the study of skew-symmetric matrices arose in the first half of the 20th century, in the works of the German mathematician Ernst Jacobsthal [1]. Now skew-symmetric matrices find applications in various fields, including statistical analysis, signal processing, matrix games and machine learning. Special methods and algorithms for solving linear systems with skew-symmetric matrices have been developed (see [2–5], for instance).

The Moore–Penrose inverse of skew-symmetric matrices is of certain interest. We are talking about generalized inversion, since skew-symmetric matrices of odd order are singular, and they may be singular also when their order is even (see [6, 7], for instance). Recall that for a real  $m \times n$  matrix  $A$  the Moore–Penrose inverse  $A^+$  is the unique  $n \times m$  matrix that satisfies the following four conditions [8]:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (A^+A)^T = A^+A, \quad (AA^+)^T = AA^+. \quad (1)$$

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If  $A$  is a square nonsingular matrix, then  $A^+ = A^{-1}$ . In this sense the Moore–Penrose inverse generalizes the ordinary matrix inversion. Note that the Moore–Penrose inverse of a skew-symmetric matrix is also skew-symmetric. Indeed, using the well-known properties of the Moore–Penrose inverse [6, 8], we have

$$(A^+)^T = (A^T)^+ = (-A)^+ = -A^+.$$

Let us consider a tridiagonal matrix

$$A = \begin{bmatrix} 0 & a_1 & & & \\ -a_1 & 0 & a_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -a_{n-2} & 0 & a_{n-1} \\ & & & -a_{n-1} & 0 \end{bmatrix}, \quad (2)$$

where  $n \geq 3$ . We assume that  $a_i \neq 0$  for all  $i = 1, 2, \dots, n-1$ . This requirement is not restrictive, since if some of the overdiagonal elements of  $A$  are equal to zero, then we can represent this matrix in a block-diagonal form

$$A = \text{diag}(A_1, A_2, \dots, A_l),$$

where the diagonal blocks  $A_i$ ,  $1 \leq i \leq l$ , are either skew-symmetric matrices with nonzero overdiagonal elements or zero blocks. In this case

$$A^+ = \text{diag}(A_1^+, A_2^+, \dots, A_l^+).$$

We will consider separately the matrices of even and odd orders.

**The Matrix of Even Order.** Let  $n = 2m$ . Then from (2) we have

$$A = \begin{bmatrix} 0 & a_1 & & & \\ -a_1 & 0 & a_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & -a_{2m-2} & 0 & a_{2m-1} \\ & & & -a_{2m-1} & 0 \end{bmatrix}. \quad (3)$$

According to the above assumption about the overdiagonal elements, this matrix is nonsingular. This statement we conclude from the easily proven equality

$$\det A = (a_1 a_3 \cdots a_{2m-1})^2. \quad (4)$$

**Remark 1.** *The requirement for the overdiagonal elements of this matrix can be somewhat relaxed. It suffices that only the overdiagonal elements in the odd rows of the matrix (3) be nonzero. This obviously follows from formula (4). However, for the sake of clarity of presentation, in the future we will adhere to the above requirement.*

Thus, in our case  $A^+ = A^{-1}$ . Let  $A^{-1} = [z_{ij}]_{2m \times 2m}$ . To find the  $j$ th column  $z^{(j)} \equiv [z_{1j} z_{2j} \cdots z_{2m,j}]^T$  of the matrix  $A^{-1}$ , we solve the system

$$Az^{(j)} = \delta^{(j)}, \quad (5)$$

where  $\delta^{(j)} \equiv [0 \dots 010 \dots 0]^T$  (the unit is located on the  $j$ th place). Let us write the system (5) in expanded form:

$$\begin{aligned}
a_1 z_{2j} &= 0, \\
-a_1 z_{1j} + a_2 z_{3j} &= 0, \\
&\vdots \\
-a_{j-2} z_{j-2j} + a_{j-1} z_{jj} &= 0, \\
-a_{j-1} z_{j-1j} + a_j z_{j+1j} &= 1, \\
-a_j z_{jj} + a_{j+1} z_{j+2j} &= 0, \\
&\vdots \\
-a_{2m-2} z_{2m-2j} + a_{2m-1} z_{2mj} &= 0, \\
-a_{2m-1} z_{2m-1j} &= 0.
\end{aligned} \tag{6}$$

Consider separately even and odd columns of the matrix  $A^{-1}$ .

**Case**  $j = 2k - 1$ ,  $1 \leq k \leq m$ .

System (6) is written as follows:

$$\begin{aligned}
a_1 z_{22k-1} &= 0, \\
-a_1 z_{12k-1} + a_2 z_{32k-1} &= 0, \\
&\vdots \\
-a_{2k-3} z_{2k-32k-1} + a_{2k-2} z_{2k-12k-1} &= 0, \\
-a_{2k-2} z_{2k-22k-1} + a_{2k-1} z_{2k2k-1} &= 1, \\
-a_{2k-1} z_{2k-12k-1} + a_{2k} z_{2k+12k-1} &= 0, \\
&\vdots \\
-a_{2m-2} z_{2m-22k-1} + a_{2m-1} z_{2m2k-1} &= 0, \\
-a_{2m-1} z_{2m-12k-1} &= 0.
\end{aligned} \tag{7}$$

It is easy to see that system (7) splits into two independent subsystems:

$$\begin{aligned}
-a_{2i-1} z_{2i-12k-1} + a_{2i} z_{2i+12k-1} &= 0, \quad i = 1, 2, \dots, m-1, \\
-a_{2m-1} z_{2m-12k-1} &= 0
\end{aligned}$$

with obvious solution  $z_{2i-12k-1} = 0$ ,  $i = 1, 2, \dots, m$  and

$$\begin{aligned}
a_1 z_{22k-1} &= 0, \\
-a_{2i} z_{2i2k-1} + a_{2i+1} z_{2i+22k-1} &= 0, \quad i = 1, 2, \dots, k-2, \\
-a_{2k-2} z_{2k-22k-1} + a_{2k-1} z_{2k2k-1} &= 1, \\
-a_{2i} z_{2i2k-1} + a_{2i+1} z_{2i+22k-1} &= 0, \quad i = k, k+1, \dots, m-1,
\end{aligned}$$

whose solution is

$$z_{2i2k-1} = \begin{cases} 0, & i = 1, 2, \dots, k-1, \\ \frac{1}{a_{2k-1}} \prod_{s=k}^{i-1} \rho_s, & i = k, k+1, \dots, m, \end{cases} \tag{8}$$

where

$$\rho_s \equiv \frac{a_{2s}}{a_{2s+1}}, \quad s = 1, 2, \dots, m-1. \tag{9}$$

**Case**  $j = 2k$ ,  $1 \leq k \leq m$ .

System (6) takes the following form:

$$\begin{aligned}
a_1 z_{22k} &= 0, \\
-a_1 z_{12k} + a_2 z_{32k} &= 0, \\
&\vdots \\
-a_{2k-2} z_{2k-22k} + a_{2k-1} z_{2k2k} &= 0, \\
-a_{2k-1} z_{2k-12k} + a_{2k} z_{2k+12k} &= 1, \\
-a_{2k} z_{2k2k} + a_{2k+1} z_{2k+22k} &= 0, \\
&\vdots \\
-a_{2m-2} z_{2m-22k} + a_{2m-1} z_{2m2k} &= 0, \\
-a_{2m-1} z_{2m-12k} &= 0.
\end{aligned} \tag{10}$$

As in the previous case, system (10) is divided into two independent subsystems:

$$\begin{aligned}
a_1 z_{22k} &= 0, \\
-a_{2i} z_{2i2k} + a_{2i+1} z_{2i+22k} &= 0, \quad i = 1, 2, \dots, m-1
\end{aligned}$$

with zero solution  $z_{2i2k} = 0$ ,  $i = 1, 2, \dots, m$ , and

$$\begin{aligned}
-a_{2i-1} z_{2i-12k} + a_{2i} z_{2i+12k} &= 0, \quad i = 1, 2, \dots, k-1, \\
-a_{2k-1} z_{2k-12k} + a_{2k} z_{2k+12k} &= 1, \\
-a_{2i-1} z_{2i-12k} + a_{2i} z_{2i+12k} &= 0, \quad i = k+1, k+2, \dots, m-1, \\
-a_{2m-1} z_{2m-12k} &= 0,
\end{aligned}$$

the solution of which is

$$z_{2i-12k} = \begin{cases} -\frac{1}{a_{2k-1}} \prod_{s=i}^{k-1} r_s, & i = 1, 2, \dots, k, \\ 0, & i = k+1, k+2, \dots, m, \end{cases} \tag{11}$$

where

$$r_s \equiv \frac{a_{2s}}{a_{2s-1}}, \quad s = 1, 2, \dots, m-1. \tag{12}$$

Summarizing the above considerations, i.e. having formulas (8) and (11), we arrive at the statement below.

**Theorem.** *Nonzero elements of the matrix  $A^{-1} = [z_{ij}]_{2m \times 2m}$  are as follows:*

$$\left. \begin{aligned} z_{2i2k-1} &= \frac{1}{a_{2k-1}} \prod_{s=k}^{i-1} \rho_s, \quad i = k, k+1, \dots, m, \\ z_{2i-12k} &= -\frac{1}{a_{2k-1}} \prod_{s=i}^{k-1} r_s, \quad i = 1, 2, \dots, k, \end{aligned} \right\}, \quad k = 1, 2, \dots, m, \tag{13}$$

where the quantities  $\rho_s$  and  $r_s$  are defined in (9) and (12), respectively.

**Example.** Below we give a skew-symmetric matrix  $A$  and the inverse matrix  $A^{-1}$ , computed by formulas (13):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

**Remark 2.** Taking into account the fact that the matrix  $A^{-1}$  is also skew-symmetric, we can halve the amount of calculations in formulas (13) by setting

$$z_{2i-1,2k} = -z_{2k,2i-1}, \quad i = 1, 2, \dots, k, \quad k = 1, 2, \dots, m. \quad (14)$$

Based on the above closed form expressions (13), let us write a computational procedure to calculate the elements of the inverse matrix  $A^{-1} = [z_{ij}]_{2m \times 2m}$  for skew-symmetric matrix  $A$  given in (3).

**Procedure MPIinverse/even order.**

1. Input elements  $a_1, a_2, \dots, a_{2m-1}$  of the matrix  $A$  (see (3)).
2. Calculate the quantities  $\rho_s$  (see (9)):

$$\rho_s = a_{2s}/a_{2s+1}, \quad s = 1, 2, \dots, m-1.$$

3. Set  $z_{ij} = 0$  for  $i, j = 1, 2, \dots, 2m$ .
4. Calculate nonzero elements from the lower triangular part of  $A^{-1}$  (see (13)):

$$\left. \begin{aligned} z_{2k,2k-1} &= 1/a_{2k-1} \\ z_{2i+2,2k-1} &= z_{2i,2k-1} \rho_i, \quad i = k, k+1, \dots, m-1 \end{aligned} \right\}, \quad k = 1, 2, \dots, m.$$

5. Set nonzero elements from the upper triangular part of  $A^{-1}$  (see (14)):

$$z_{2i-1,2k} = -z_{2k,2i-1}, \quad i = 1, 2, \dots, k, \quad k = 1, 2, \dots, m.$$

6. Output the matrix  $A^{-1} = [z_{ij}]_{2m \times 2m}$ .

**End Procedure.**

In conclusion, a few words about the computational complexity of the procedure **MPIinverse/even order**. A simple calculations show that computing all nonzero elements of the lower triangular part of the matrix  $A^{-1}$  requires  $0.5m^2 + O(m)$  multiplications. Note that the number of mentioned nonzero elements is  $m(m+1)/2$ .

**The Matrix of Odd Order.** Let  $n = 2m + 1$ . From (2) we have

$$A = \begin{bmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & a_2 & & & 0 \\ & \ddots & \ddots & \ddots & & \\ 0 & & -a_{2m-1} & 0 & a_{2m} & \\ & & & -a_{2m} & 0 & \end{bmatrix}. \quad (15)$$

Regardless of the values of the overdiagonal elements, this matrix is singular. In the present work, the computation of the Moore–Penrose inverse  $A^+$  is based on a special notation of the matrix  $A$ .

Let us introduce bidiagonal matrix

$$B = \begin{bmatrix} a_1 & -a_2 & & & \\ & a_3 & -a_4 & 0 & \\ & 0 & \ddots & \ddots & \\ & & & a_{2m-1} & -a_{2m} \end{bmatrix} \quad (16)$$

of the size  $m \times m + 1$ . Next, we define the following matrices:

$$P = [p_{ij}]_{2m+1 \times m}, \quad p_{ij} = \begin{cases} 1, & \text{if } i = 2j, \\ 0, & \text{if } i \neq 2j, \end{cases} \quad j = 1, 2, \dots, m, \quad (17)$$

and

$$Q = [q_{ij}]_{m+1 \times 2m+1}, \quad q_{ij} = \begin{cases} 1, & \text{if } j = 2i - 1, \\ 0, & \text{if } j \neq 2i - 1, \end{cases} \quad i = 1, 2, \dots, m + 1. \quad (18)$$

Then the matrix  $A$  can be written as follows:

$$A = (PBQ)^T - PBQ. \quad (19)$$

The following easily verified properties of the matrices  $P$  and  $Q$  hold:

$$P^T P = I_m, \quad QQ^T = I_{m+1}, \quad QP = 0, \quad PP^T + Q^T Q = I_{2m+1} \quad (20)$$

(the subscript of the identity matrix indicates its order).

**Lemma.** *Let the matrix  $A$ , defined in (15), be written in the form (19). Then*

$$A^+ = (Q^T B^+ P^T)^T - Q^T B^+ P^T. \quad (21)$$

*Proof.* It suffices to show that matrix (21) satisfies all four conditions (1). We will use properties (20) of the matrices  $P$  and  $Q$ . Let us first establish third and fourth of conditions (1). We have

$$\begin{aligned} A^+ A &= [(Q^T B^+ P^T)^T - Q^T B^+ P^T][(PBQ)^T - PBQ] \\ &= P(B^+)^T (QQ^T) B^T P^T - P(B^+)^T (QP) BQ \\ &\quad - Q^T B^+ (QP)^T B^T P^T + Q^T B^+ (P^T P) BQ \\ &= P(BB^+)^T P^T + Q^T (B^+ B) Q. \end{aligned} \quad (22)$$

From this it obviously follows that  $(A^+ A)^T = A^+ A$ . Similarly,

$$\begin{aligned} AA^+ &= [(PBQ)^T - PBQ][(Q^T B^+ P^T)^T - Q^T B^+ P^T] \\ &= Q^T B^T (P^T P) (B^+)^T Q - Q^T B^T (QP)^T B^+ P^T \\ &\quad - PB(QP)(B^+)^T Q + PB(QQ^T) B^+ P^T \\ &= Q^T (B^+ B)^T Q + P(BB^+) P^T, \end{aligned}$$

which implies that  $(AA^+)^T = AA^+$ .

Let us now turn to the first two conditions from (1). Taking into account equality (22), we obtain:

$$\begin{aligned} AA^+ A &= [(PBQ)^T - PBQ][P(BB^+)^T P^T + Q^T (B^+ B) Q] \\ &= Q^T B^T (P^T P) (BB^+)^T P^T + Q^T B^T (QP)^T (B^+ B) Q \\ &\quad - PB(QP)(BB^+)^T P^T - PB(QQ^T) (B^+ B) Q \\ &= Q^T (BB^+ B)^T P^T - P(BB^+ B) Q = Q^T B^T P^T - PBQ = A. \end{aligned}$$

Further,

$$\begin{aligned} A^+AA^+ &= [P(BB^+)^T P^T + Q^T(B^+B)Q][(Q^T B^+ P^T)^T - Q^T B^+ P^T] \\ &= P(BB^+)^T (P^T P)(B^+)^T Q - P(BB^+)^T (QP)^T B^+ P^T \\ &\quad + Q^T (B^+B)(QP)(B^+)^T Q - Q^T (B^+B)(QQ^T)B^+ P^T \\ &= P(B^+BB^+)^T Q - Q^T (B^+BB^+)P^T = P(B^+)^T Q - Q^T B^+ P^T = A^+. \end{aligned}$$

The validity of representation (21) for  $A^+$  has been proved.  $\square$

So, the problem of finding the Moore–Penrose inverse for the matrix  $A$  given in (15) is reduced to a similar problem for the matrix  $B$  defined in (16).

**A Way of Computing the Matrix  $B^+$ .** An approach to derive the Moore–Penrose inverse of the matrix  $B$  is based upon the well-known formula

$$B^+ = \lim_{\varepsilon \rightarrow +0} (B^T B + \varepsilon I_{m+1})^{-1} B^T \quad (23)$$

(see [8], for instance). Our plan is as follows. First we find the inverse matrix  $(B^T B + \varepsilon I_{m+1})^{-1}$ . Further, the elements of the matrix  $(B^T B + \varepsilon I_{m+1})^{-1} B^T$  are calculated and a character of their dependence on the parameter  $\varepsilon$  is revealed. Thereafter, according to equality (23), passing to the limit as  $\varepsilon \rightarrow +0$ , we will arrive at closed form expressions for the elements of the matrix  $B^+$ .

This study will be carried out in the subsequent, second part of this work.

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ԵՐԵՔԱՆԿՅՈՒՆԱԳԾԱՅԻՆ ՇԵՂԱՍԻՄԵՏՐԻԿ ՄԱՏՐԻՑՆԵՐԻ  
ՄՈՒՐ-ՊԵՆՐՈՒԶԻ ՆԱԿԱՂԱՐՁՈՒՄԸ. I

Սույն աշխատանքը նվիրված է Մուր-Պենրոուզի հակադարձ մատրիցի փարրերի բացահայտ արտահայտությունների արպածմանը՝ երեքանկյունագծային իրական շեղասիմետրիկ մատրիցների համար: Աշխատանքի առաջին մասում սրացված են արդյունքներ զույգ կարգի մատրիցների համար: Բացի այդ, կենր կարգի մատրիցների համար նշված է ընդհանրացված հակադարձի հաշվման ճանապարհը:

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ОБРАЩЕНИЕ МУРА-ПЕНРОУЗА ТРЕХДИАГОНАЛЬНЫХ  
КОСОСИММЕТРИЧНЫХ МАТРИЦ. I

Настоящая работа посвящена выводу явных выражений для элементов обратной матрицы Мура-Пенроуза в случае трехдиагональных вещественных кососимметричных матриц. В первой части работы получены результаты для матриц четного порядка. Кроме того, намечен путь вычисления обобщенного обращения для матриц нечетного порядка.