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Mathematics

THE MOORE–PENROSE INVERSE OF TRIDIAGONAL SKEW-SYMMETRIC MATRICES. II

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This article is the second part of the work started in the previous publication by the authors [1]. The results presented here relate to deriving closed form expressions for the elements of the Moore–Penrose inverse of tridiagonal real skew-symmetric matrices of odd order. On the base of the formulas obtained, an algorithm that is optimal in terms of the amount of computational efforts is constructed.

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Preliminaries. Let us continue the study of skew-symmetric matrices of odd order, begun in the article [1]. Consider a skew-symmetric matrix

$$A = \begin{bmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & a_2 & & & 0 \\ & \ddots & \ddots & & \ddots & \\ 0 & & -a_{2m-1} & & 0 & a_{2m} \\ & & & & -a_{2m} & 0 \end{bmatrix}, \tag{1}$$

where $a_i \neq 0,\quad i = 1,2,\ldots,2m$. Recall some points already noted in [1]. There we introduced bidiagonal matrix

$$B = \begin{bmatrix} a_1 & -a_2 & & & \\ & a_3 & -a_4 & 0 & \\ & 0 & \ddots & \ddots & \\ & & & a_{2m-1} & -a_{2m} \end{bmatrix} \tag{2}$$

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of the size $m \times m+1$, as well as matrices

$$P = [p_{ij}]_{2m+1\times m}, \quad p_{ij} = \begin{cases} 1, & \text{if} \quad i = 2j, \\ 0, & \text{if} \quad i \neq 2j, \end{cases} \quad j = 1,2,\dots,m, \tag{3}$$

and

$$Q = [q_{ij}]_{m+1\times 2m+1}, \quad q_{ij} = \begin{cases} 1, & \text{if} \quad j = 2i-1, \\ 0, & \text{if} \quad j \neq 2i-1 \end{cases} \quad i = 1,2,\dots,m+1. \tag{4}$$

It was shown that the matrix A can be written in the form

$$A = (PBQ)^T - PBQ. \tag{5}$$

Then, as it was proved (see Lemma 1 in [1]), the matrix A^+ is as follows:

$$A^+ = (Q^T B^+ P^T)^T - Q^T B^+ P^T. \tag{6}$$

Thus the problem of finding the Moore–Penrose inverse for the matrix A from (1) is reduced to a similar problem for the matrix B given in (2).

To calculate the Moore–Penrose inverse of the matrix B, we use the following well-known formula:

$$B^+ = \lim_{\varepsilon\to+0} (B^T B + \varepsilon I_{m+1})^{-1} B^T, \tag{7}$$

where I_{m+1} is the identity matrix of order $m+1$ (see [2], for instance). First we find the inverse matrix $(B^T B + \varepsilon I_{m+1})^{-1}$. Next, we reveal how the elements of the matrix $(B^T B + \varepsilon I_{m+1})^{-1} B^T$ depend on the parameter ε. Then, according to the equality (7), passing to the limit as $\varepsilon \to +0$, we arrive at closed form expressions for the elements of the matrix B^+.

Thus the first problem that arise is the inversion of nonsingular tridiagonal symmetric matrix

$$B^T B + \varepsilon I_{m+1} =$$

$$\begin{bmatrix} a_1^2+\varepsilon & -a_1a_2 & & & & \\ -a_1a_2 & a_2^2+a_3^2+\varepsilon & -a_3a_4 & & & 0 \\ & \ddots & \ddots & & \ddots & \\ 0 & & -a_{2m-3}a_{2m-2} & & a_{2m-2}^2+a_{2m-1}^2+\varepsilon & -a_{2m-1}a_{2m} \\ & & & & -a_{2m-1}a_{2m} & a_{2m}^2+\varepsilon \end{bmatrix}. \tag{8}$$

To obtain the inverse matrix $(B^T B + \varepsilon I_{m+1})^{-1} = [x_{ij}(\varepsilon)]_{m+1\times m+1}$, let us take advantage of a computational procedure developed in [3]. As applied to matrix (8), this procedure is as follows.

Procedure Inverse $(B^T B + \varepsilon I_{m+1})^{-1}$**.**

1. Calculate quantities f_i, g_i, p and q:

$$f_i = -\frac{a_{2i-2}^2 + a_{2i-1}^2 + \varepsilon}{a_{2i-3}a_{2i-2}}, \quad i = 2,3,\dots,m, \tag{9}$$

$$g_i = \frac{a_{2i-1}a_{2i}}{a_{2i-3}a_{2i-2}}, \quad i = 2,3,\dots,m, \tag{10}$$

$$p = -\frac{a_1^2+\varepsilon}{a_1 a_2}, \quad q = -\frac{a_{2m}^2+\varepsilon}{a_{2m-1}a_{2m}}. \tag{11}$$

2. Calculate recursively quantities μ_i :

$$\begin{aligned} &\mu_{m+1} = 1, \quad \mu_m = -q, \\ &\mu_i = -f_{i+1}\mu_{i+1} - g_{i+1}\mu_{i+2}, \quad i = m-1, m-2, \ldots, 1. \end{aligned} \tag{12}$$

3. Calculate recursively quantities ν_i :

$$\begin{aligned} &\nu_1 = 1, \quad \nu_2 = -p, \\ &\nu_i = -(\nu_{i-2} + f_{i-1}\nu_{i-1})/g_{i-1}, \quad i = 3, 4, \ldots, m+1. \end{aligned} \tag{13}$$

4. Calculate quantity t :

$$t = [(a_1^2+\varepsilon)\mu_1 - a_1 a_2 \mu_2]^{-1}. \tag{14}$$

5. Calculate the upper triangular part of the matrix $(B^T B + \varepsilon I_{m+1})^{-1}$:

$$x_{ij}(\varepsilon) = t\mu_j \nu_i, \quad i = 1, 2, \ldots, j, \quad j = 1, 2, \ldots, m+1\,. \tag{15}$$

6. Set the lower triangular part of the matrix $(B^T B + \varepsilon I_{m+1})^{-1}$:

$$x_{ij}(\varepsilon) = x_{ji}(\varepsilon), \quad i = j+1, j+2, \ldots, m+1, \quad j = 1, 2, \ldots, m. \tag{16}$$

End Procedure.

In accordance with our plan, let us carry out a more detailed study of the quantities successively computed in the procedure **Inverse** $(B^T B + \varepsilon I_{m+1})^{-1}$. We are interested in how these quantities depend on the parameter ε.

Some Auxiliary Formulas and Relations.

• *The quantities f_i $(2 \le i \le m)$, p and q from (9) and (11).*

We set

$$f_i = \overset{\circ}{f}_i + \alpha_i\varepsilon, \ \ i = 2, 3, \ldots, m, \ \text{ where } \overset{\circ}{f}_i = -\frac{a_{2i-2}^2 + a_{2i-1}^2}{a_{2i-3}a_{2i-2}}, \ \ \alpha_i = -\frac{1}{a_{2i-3}a_{2i-2}}, \tag{17}$$

$$p = \overset{\circ}{p} + \alpha_1\varepsilon, \ \text{ where } \overset{\circ}{p} = -\frac{a_1}{a_2}, \ \ \alpha_1 = -\frac{1}{a_1 a_2}, \tag{18}$$

$$q = \overset{\circ}{q} + \alpha_{m+1}\varepsilon, \ \text{ where } \overset{\circ}{q} = -\frac{a_{2m}}{a_{2m-1}}, \ \ \alpha_{m+1} = -\frac{1}{a_{2m-1}a_{2m}}. \tag{19}$$

• *The quantities μ_i $(1 \le i \le m+1)$ from (12).*

L e m m a 1. *The quantities μ_i can be represented as*

$$\begin{aligned} &\mu_{m+1} = \overset{\circ}{\mu}_{m+1} + \gamma_{m+1}\varepsilon, \quad \mu_m = \overset{\circ}{\mu}_m + \gamma_m\varepsilon, \\ &\mu_i = \overset{\circ}{\mu}_i + \gamma_i\varepsilon + O(\varepsilon^2), \quad 1 \le i \le m-1, \end{aligned} \tag{20}$$

where the quantities $\overset{\circ}{\mu}_i$ and γ_i satisfy the following recurrence relations:

$$\begin{aligned} &\overset{\circ}{\mu}_{m+1} = 1, \quad \overset{\circ}{\mu}_m = -\overset{\circ}{q}, \\ &\overset{\circ}{\mu}_i = -\overset{\circ}{f}_{i+1}\overset{\circ}{\mu}_{i+1} - g_{i+1}\overset{\circ}{\mu}_{i+2}, \quad i = m-1, m-2, \ldots, 1, \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\gamma_{m+1} = 0, \quad \gamma_m = -\alpha_{m+1}, \\ &\gamma_i = -\overset{\circ}{f}_{i+1}\gamma_{i+1} - g_{i+1}\gamma_{i+2} - \alpha_{i+1}\overset{\circ}{\mu}_{i+1}, \quad i = m-1, m-2, \ldots, 1. \end{aligned} \tag{22}$$

Proof. Since $\mu_{m+1} = 1$ (see (12)), then we set $\overset{\circ}{\mu}_{m+1} = 1$, $\gamma_{m+1} = 0$. Further, $\mu_m = -q$. According to (19), we can take $\overset{\circ}{\mu}_m = -\overset{\circ}{q}$, $\gamma_m = -\alpha_{m+1}$. For indices in the range $1 \le i \le m-1$, required representations (20) and recurrence relations (21), (22) can be readily derived by induction from relations (12), taking into account (17). □

Let us introduce the notation

$$r_s \equiv \frac{a_{2s}}{a_{2s-1}}, \quad s = 1,2,\dots,m. \tag{23}$$

Additionally, we set $r_0 = r_{m+1} = 1$.

The quantities $\overset{\circ}{\mu}_i$ computed by recursion (21), can be written in a closed form.

Lemma 2. *The following equalities hold:*

$$\overset{\circ}{\mu}_i = \prod_{s=i}^{m} r_s, \quad i = 1,2,\dots,m+1. \tag{24}$$

Proof. First, the value $\overset{\circ}{\mu}_{m+1} = 1$ corresponds to the record (24). Then, as follows from (21) and (19),

$$\overset{\circ}{\mu}_m = -\overset{\circ}{q} = \frac{a_{2m}}{a_{2m-1}} = r_m .$$

Further reasonings are carried out by induction. Using expressions (10), (17) and proceeding from (21), for the values $i = m-1, m-2, \dots, 1$ we get that

$$\begin{aligned}\overset{\circ}{\mu}_i &= \frac{a_{2i}^2 + a_{2i+1}^2}{a_{2i-1}a_{2i}} \prod_{s=i+1}^{m} r_s - \frac{a_{2i+1}a_{2i+2}}{a_{2i-1}a_{2i}} \prod_{s=i+2}^{m} r_s \\ &= \left(\frac{a_{2i}^2 + a_{2i+1}^2}{a_{2i-1}a_{2i}} - \frac{a_{2i+1}a_{2i+2}}{a_{2i-1}a_{2i}} \cdot \frac{1}{r_{i+1}} \right) \prod_{s=i+1}^{m} r_s = r_i \prod_{s=i+1}^{m} r_s = \prod_{s=i}^{m} r_s,\end{aligned}$$

which completes the proof. □

The next assertion is a simple consequence of formula (24).

Corollary 1. *The following relations hold:*

$$\overset{\circ}{\mu}_i = r_i \overset{\circ}{\mu}_{i+1}, \quad i = m, m-1, \dots, 1. \tag{25}$$

Below we need some formulas related to the quantities γ_i, $1 \le i \le m+1$, defined in (22). Consider the quantities

$$R_i \equiv \gamma_i - r_i \gamma_{i+1}, \quad i = 1,2,\dots,m. \tag{26}$$

Lemma 3. *The following relations hold:*

$$R_m = \frac{r_m}{a_{2m}^2}, \quad R_i = \frac{r_i}{a_{2i}^2} \left(a_{2i+1}^2 R_{i+1} + \overset{\circ}{\mu}_{i+1} \right), \quad i = m-1, m-2, \dots, 1. \tag{27}$$

Proof. As follows from (22) and (19),

$$R_m = \gamma_m - r_m \gamma_{m+1} = \gamma_m = -\alpha_{m+1} = \frac{r_m}{a_{2m}^2}. \tag{28}$$

Next, from relation (22) (for $i = m-1$), taking into account equality (28), we have

$$\begin{aligned} R_{m-1} &= \gamma_{m-1} - r_{m-1}\gamma_m = -(\overset{\circ}{f}_m + r_{m-1})\gamma_m - \alpha_m \overset{\circ}{\mu}_m \\ &= -\alpha_m \left(\frac{\overset{\circ}{f}_m + r_{m-1}}{\alpha_m} R_m + \overset{\circ}{\mu}_m \right) = \frac{r_{m-1}}{a_{2m-2}^2} \left(a_{2m-1}^2 R_m + \overset{\circ}{\mu}_m \right), \end{aligned}$$

that is,

$$R_{m-1} = \frac{r_{m-1}}{a_{2m-2}^2} \left(a_{2m-1}^2 R_m + \overset{\circ}{\mu}_m \right). \tag{29}$$

Let $1 \le i \le m-2$. From relations (22), using expressions (10) and (17), we obtain

$$\begin{aligned} R_i &= \gamma_i - r_i\gamma_{i+1} = -\overset{\circ}{f}_{i+1} \gamma_{i+1} - g_{i+1}\gamma_{i+2} - \alpha_{i+1} \overset{\circ}{\mu}_{i+1} \\ &= \frac{1}{a_{2i}^2} \left[r_i a_{2i+1}^2 (\gamma_{i+1} - r_{i+1}\gamma_{i+2}) + r_i \overset{\circ}{\mu}_{i+1} \right] = \frac{r_i}{a_{2i}^2} \left(a_{2i+1}^2 R_{i+1} + \overset{\circ}{\mu}_{i+1} \right). \end{aligned}$$

Thus, in conjunction with (28) and (29), we arrive to the relations (27). □

Lemma 4. *For the quantities R_i defined in (26), the expressions*

$$R_i = \frac{r_i}{a_{2i}^2} \cdot \sum_{k=i}^{m} \left(\prod_{s=i+1}^{k} \frac{1}{r_s} \right) \left(\prod_{s=k+1}^{m} r_s \right), \quad i = 1, 2, \ldots, m \tag{30}$$

are valid.

Proof. It is easy to verify the validity of the assertion by substituting expressions (30) into recurrence relations (27). □

- *The quantities ν_i ($1 \le i \le m+1$) from (13).*

Lemma 5. *The quantities ν_i can be represented as*

$$\begin{aligned} &\nu_1 = \overset{\circ}{\nu}_1 + \delta_1\varepsilon, \quad \nu_2 = \overset{\circ}{\nu}_2 + \delta_2\varepsilon, \\ &\nu_i = \overset{\circ}{\nu}_i + \delta_i\varepsilon + O(\varepsilon^2), \quad 1 \le i \le m-1, \end{aligned} \tag{31}$$

where the quantities $\overset{\circ}{\nu}_i$ and δ_i satisfy the following recurrence relations:

$$\begin{aligned} &\overset{\circ}{\nu}_1 = 1, \quad \overset{\circ}{\nu}_2 = -\overset{\circ}{p}, \\ &\overset{\circ}{\nu}_i = -(\overset{\circ}{f}_{i-1} \overset{\circ}{\nu}_{i-1} + \overset{\circ}{\nu}_{i-2})/g_{i-1}, \quad i = 3, 4, \ldots, m+1, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\delta_1 = 0, \ \delta_2 = -\alpha_1, \\ &\delta_i = -(\overset{\circ}{f}_{i-1} \delta_{i-1} + \delta_{i-2} + \alpha_{i-1} \overset{\circ}{\nu}_{i-1})/g_{i-1}, \quad i = 3, 4, \ldots, m+1. \end{aligned} \tag{33}$$

Proof. Since $\nu_1 = 1$ (see (13)) then we set $\overset{\circ}{\nu}_1 = 1$, $\delta_1 = 0$. Further, $\nu_2 = -p$. According to (18), we take $\overset{\circ}{\nu}_2 = -\overset{\circ}{p}$, $\delta_2 = -\alpha_1$. For indices in the range $1 \le i \le m-1$, required representations (31) and recurrence relations (32),(33) are easily derived by induction from the relations (13), taking into account (17). □

The quantities $\overset{\circ}{v}_i$ computed by recursion (32), can be written in a closed form.

Lemma 6. *The following equalities hold:*

$$\overset{\circ}{v}_i = \prod_{s=1}^{i-1} \frac{1}{r_s}, \quad i = 1,2,\ldots,m+1. \tag{34}$$

Proof. Note, that the value $\overset{\circ}{v}_1 = 1$ conforms to record (34). Next, as follows from (32) and (18),

$$\overset{\circ}{v}_2 = -\overset{\circ}{p} = \frac{a_1}{a_2} = \frac{1}{r_1}.$$

Further reasonings are carried out by induction. Proceeding from (32) and using expressions (10), (17), for the values $i = 3,4,\ldots,m+1$ we obtain that

$$\overset{\circ}{v}_i = \frac{a_{2i-4}^2 + a_{2i-3}^2}{a_{2i-3}a_{2i-2}} \prod_{s=1}^{i-2} \frac{1}{r_s} - \frac{a_{2i-5}a_{2i-4}}{a_{2i-3}a_{2i-2}} \prod_{s=1}^{i-3} \frac{1}{r_s}$$

$$= \left(\frac{a_{2i-4}^2 + a_{2i-3}^2}{a_{2i-3}a_{2i-2}} - \frac{a_{2i-5}a_{2i-4}}{a_{2i-3}a_{2i-2}} r_{i-2} \right) \prod_{s=1}^{i-2} \frac{1}{r_s} = \frac{1}{r_{i-1}} \prod_{s=1}^{i-2} \frac{1}{r_s} = \prod_{s=1}^{i-1} \frac{1}{r_s}$$

□

The next statement is a simple consequence of the formula (34).

Corollary 2. *The following relations hold:*

$$\overset{\circ}{v}_{i+1} = \frac{1}{r_i} \overset{\circ}{v}_i, \quad i = 1,2,\ldots,m. \tag{35}$$

Let us deduce some formulas related to the quantities δ_i, $1 \le i \le m+1$, defined in (33). Consider the quantities

$$S_i \equiv r_i \delta_{i+1} - \delta_i, \quad i = 1,2,\ldots,m. \tag{36}$$

Lemma 7. *The following relations hold:*

$$S_1 = \frac{1}{a_1^2}, \quad S_i = \frac{1}{a_{2i-1}^2} \cdot \frac{1}{r_{i-1}} \left(a_{2i-2}^2 S_{i-1} + \overset{\circ}{v}_{i-1} \right), \quad i = 2,3,\ldots,m. \tag{37}$$

Proof. From (33) and (18) we obtain, that

$$S_1 = r_1\delta_2 - \delta_1 = r_1\delta_2 = -r_1\alpha_1 = \frac{1}{a_1^2}. \tag{38}$$

Further, from relation (33) (for $i = 3$), taking into account expressions (10), (17) and equality (38), we have

$$\begin{aligned} S_2 &= -\left(1 + \frac{r_2 \overset{\circ}{f}_2}{g_2}\right)\delta_2 - \frac{r_2\alpha_2}{g_2} \overset{\circ}{v}_2 = \frac{1}{r_1}\left[-\left(1 + \frac{r_2 \overset{\circ}{f}_2}{g_2}\right) S_1 - \frac{r_2\alpha_2}{g_2} \overset{\circ}{v}_1\right] \\ &= \frac{1}{r_1}\left(\frac{a_2^2}{a_3^2} S_1 + \frac{1}{a_3^2} \overset{\circ}{v}_1\right) = \frac{1}{a_3^2} \cdot \frac{1}{r_1}\left(a_2^2 S_1 + \overset{\circ}{v}_1\right), \end{aligned}$$

that is,

$$S_2 = \frac{1}{a_3^2} \cdot \frac{1}{r_1}\left(a_2^2 S_1 + \overset{\circ}{v}_1\right). \tag{39}$$

Consider now indices in the range $3 \le i \le m$. From relations (33), using expressions (10) and (17), we get:

$$S_i = -r_i\left(\overset{\circ}{f}_i\,\delta_i + \delta_{i-1} + \alpha_i\,\overset{\circ}{\nu}_i\right)/g_i - \delta_i = -\left(1 + \frac{r_i\,\overset{\circ}{f}_i}{g_i}\right)\delta_i - \frac{r_i}{g_i}\delta_{i-1}$$

$$-\frac{r_i\alpha_i}{g_i}\cdot\frac{1}{r_{i-1}}\,\overset{\circ}{\nu}_{i-1} = \frac{a_{2i-2}^2}{a_{2i-1}^2}\delta_i - \frac{a_{2i-2}^2}{a_{2i-1}^2}\cdot\frac{1}{r_{i-1}}\delta_{i-1} + \frac{1}{a_{2i-1}^2}\cdot\frac{1}{r_{i-1}}\,\overset{\circ}{\nu}_{i-1}$$

$$= \frac{a_{2i-2}^2}{a_{2i-1}^2}\cdot\frac{1}{r_{i-1}}(r_{i-1}\delta_i - \delta_{i-1}) + \frac{1}{a_{2i-1}^2}\cdot\frac{1}{r_{i-1}}\,\overset{\circ}{\nu}_{i-1}$$

$$= \frac{a_{2i-2}^2}{a_{2i-1}^2}\cdot\frac{1}{r_{i-1}}S_{i-1} + \frac{1}{a_{2i-1}^2}\cdot\frac{1}{r_{i-1}}\,\overset{\circ}{\nu}_{i-1} = \frac{1}{a_{2i-1}^2}\cdot\frac{1}{r_{i-1}}(a_{2i-2}^2 S_{i-1} + \overset{\circ}{\nu}_{i-1}).$$

In conjunction with (38) and (39), we arrive to (37). □

Lemma 8. *For the quantities S_i defined in (36), the expressions*

$$S_i = \frac{r_i}{a_{2i-1}^2}\cdot\sum_{k=1}^{i}\left(\prod_{s=1}^{k-1} r_s\right)\left(\prod_{s=k}^{i}\frac{1}{r_s}\right), \quad i = 1,2,\ldots,m, \tag{40}$$

are valid.

Proof. The validity of the assertion is established by substituting expressions (40) into recurrence relations (37). □

- *The quanty t from (14).*

According to representations (20) for the quantities μ_i, we have:

$$(a_1^2+\varepsilon)\mu_1 - a_1a_2\mu_2 = a_1^2(\overset{\circ}{\mu}_1 - r_1\,\overset{\circ}{\mu}_2) + (a_1^2(\gamma_1 - r_1\gamma_2) + \overset{\circ}{\mu}_1)\varepsilon + O(\varepsilon^2).$$

Since $\overset{\circ}{\mu}_1 = r_1\,\overset{\circ}{\mu}_2$ (see (25)), then, taking into account (26),

$$(a_1^2+\varepsilon)\mu_1 - a_1a_2\mu_2 = (a_1^2R_1 + \overset{\circ}{\mu}_1)\varepsilon + O(\varepsilon^2).$$

Thus

$$t = [(a_1^2R_1 + \overset{\circ}{\mu}_1)\varepsilon + O(\varepsilon^2)]^{-1}. \tag{41}$$

Having revealed the nature of the dependence of the quantities μ_i, ν_i and t on the parameter ε, we proceed obtaining formulas for the elements of the matrix B^+.

Closed Form Representation of the Matrix B^+. Let us introduce the matrix

$$Y(\varepsilon) \equiv (B^TB + \varepsilon I_{m+1})^{-1}B^T. \tag{42}$$

Hence, by equality (7), $B^+ = \lim_{\varepsilon\to+0} Y(\varepsilon)$. If

$$Y(\varepsilon) = [y_{ij}(\varepsilon)]_{m+1\times m}, \quad B^+ = [w_{ij}]_{m+1\times m},$$

then

$$w_{ij} = \lim_{\varepsilon\to+0} y_{ij}(\varepsilon), \quad 1 \le i \le m+1,\ 1 \le j \le m. \tag{43}$$

As follows from (42) and (2), the elements of the matrix $Y(\varepsilon)$ are calculated by the rule

$$y_{ij}(\varepsilon) = x_{ij}(\varepsilon)a_{2j-1} - x_{i\,j+1}(\varepsilon)a_{2j}, \quad 1 \le i \le m+1,\ 1 \le j \le m. \tag{44}$$

Subject to expressions (15) and (16), for a fixed index j in the range $1 \le j \le m$ we will consider separately two cases: $1 \le i \le j$ and $j+1 \le i \le m+1$.

• *Indices* $1 \le i \le j$.

From (44), using expressions (15) for the elements $x_{ij}(\varepsilon)$, we write

$$y_{ij}(\varepsilon) = t\,\nu_i(\mu_j a_{2j-1} - \mu_{j+1}a_{2j}) = t\,\nu_i a_{2j-1}(\mu_j - r_j\mu_{j+1}).$$

Taking advantage of representations (20) and (31) of the quantities μ_j and ν_i, respectively, we have

$$\begin{aligned} y_{ij}(\varepsilon) &= t\,a_{2j-1}(\mathring{\nu}_i + \delta_i\varepsilon + O(\varepsilon^2))[(\mathring{\mu}_j + \gamma_j\varepsilon + O(\varepsilon^2)) - r_j(\mathring{\mu}_{j+1} + \gamma_{j+1}\varepsilon + O(\varepsilon^2))] \\ &= t\,a_{2j-1}(\mathring{\nu}_i + \delta_i\varepsilon + O(\varepsilon^2))[(\mathring{\mu}_j - r_j\,\mathring{\mu}_{j+1}) + (\gamma_j - r_j\gamma_{j+1})\varepsilon + O(\varepsilon^2)]. \end{aligned}$$

Hence, due to relation (25) and notation (26), we get the equality

$$y_{ij}(\varepsilon) = t\,a_{2j-1}\,\varepsilon\,(\mathring{\nu}_i + \delta_i\varepsilon + O(\varepsilon^2))(R_j + O(\varepsilon)).$$

Then, substituting expression (41) for the quantity t into the right hand side of the previous equality yields

$$y_{ij}(\varepsilon) = a_{2j-1}\,\frac{(\mathring{\nu}_i + \delta_i\varepsilon + O(\varepsilon^2))(R_j + O(\varepsilon))}{a_1^2R_1 + \mathring{\mu}_1 + O(\varepsilon)}.$$

According to (43), by taking limit we find

$$w_{ij} = a_{2j-1}\,\frac{\mathring{\nu}_i\,R_j}{a_1^2R_1 + \mathring{\mu}_1}. \tag{45}$$

Further, let us substitute expressions (24),(30) and (34) into the right hand side of equality (45). As a result, making simple transformations, we arrive at the expression

$$w_{ij} = \frac{\left(\prod\limits_{s=1}^{i-1}\frac{1}{r_s}\right)\sum\limits_{k=j}^{m}\left(\prod\limits_{s=j}^{k}\frac{1}{r_s}\right)\left(\prod\limits_{s=k+1}^{m} r_s\right)}{a_{2j-1}\sum\limits_{k=0}^{m}\left(\prod\limits_{s=1}^{k}\frac{1}{r_s}\right)\left(\prod\limits_{s=k+1}^{m} r_s\right)}, \quad i = 1,2,\ldots,j. \tag{46}$$

Remark. *The denominator of the fraction in the right hand side of equality (46) is nonzero. This obviously follows from the fact that all terms in the sum have the same sign.*

• *Indices* $j+1 \le i \le m+1$.

Using expressions for the elements $x_{ij}(\varepsilon)$ (see (15) and (16)), from (44) we have

$$y_{ij}(\varepsilon) = t\,\mu_i(\nu_j a_{2j-1} - \nu_{j+1}a_{2j}) = t\,a_{2j-1}\mu_i(\nu_j - r_j\nu_{j+1}).$$

It follows from representations (20) and (31) that

$$\begin{aligned} y_{ij}(\varepsilon) &= t\,a_{2j-1}(\overset{\circ}{\mu}_i + \gamma_i\varepsilon + O(\varepsilon^2))[(\overset{\circ}{\nu}_j + \delta_j\varepsilon + O(\varepsilon^2)) - r_j(\overset{\circ}{\nu}_{j+1} + \delta_{j+1}\varepsilon + O(\varepsilon^2))] \\ &= t\,a_{2j-1}(\overset{\circ}{\mu}_i + \gamma_i\varepsilon + O(\varepsilon^2))[(\overset{\circ}{\nu}_j - r_j\,\overset{\circ}{\nu}_{j+1}) + (\delta_j - r_j\delta_{j+1})\varepsilon + O(\varepsilon^2)]. \end{aligned}$$

Hence, due to relation (35) and notation (36), we obtain the following equality:

$$y_{ij}(\varepsilon) = -t\,a_{2j-1}\,\varepsilon\,(\overset{\circ}{\mu}_i + \gamma_i\varepsilon + O(\varepsilon^2))(S_j + O(\varepsilon)).$$

By substituting expression (41) into the previous equality, we get

$$y_{ij}(\varepsilon) = -a_{2j-1}\,\frac{(\overset{\circ}{\mu}_i + \gamma_i\varepsilon + O(\varepsilon^2))(S_j + O(\varepsilon))}{a_1^2 R_1 + \overset{\circ}{\mu}_1 + O(\varepsilon)}.$$

Taking limit as $\varepsilon \to +\infty$, according to (43) we find

$$w_{ij} = -a_{2j-1}\,\frac{\overset{\circ}{\mu}_i\,S_j}{a_1^2 R_1 + \overset{\circ}{\mu}_1}. \tag{47}$$

Further, substituting expressions (24), (30) and (40) into the right hand side of the equality (47) yields

$$w_{ij} = -\frac{\left(\prod_{s=i}^{m} r_s\right)\sum_{k=1}^{j}\left(\prod_{s=1}^{k-1} r_s\right)\left(\prod_{s=k}^{j-1}\frac{1}{r_s}\right)}{a_{2j-1}\sum_{k=0}^{m}\left(\prod_{s=1}^{k}\frac{1}{r_s}\right)\left(\prod_{s=k+1}^{m} r_s\right)}, \quad i = j+1, j+2, \ldots, m+1. \tag{48}$$

Combining the above considerations, i.e. having formulas (46) and (48), we arrive to the following statement.

Theorem 1. *Let B be an $m\times m+1$ bidiagonal matrix given in (2) and $a_i \neq 0$, $1 \le i \le 2m$. Then the elements of the Moore–Penrose inverse $B^+ = [w_{ij}]_{m+1\times m}$ are as follows: for $1 \le j \le m$,*

$$w_{ij} = \frac{\left(\prod_{s=1}^{i-1}\frac{1}{r_s}\right)\sum_{k=j}^{m}\left(\prod_{s=j}^{k}\frac{1}{r_s}\right)\left(\prod_{s=k+1}^{m} r_s\right)}{a_{2j-1}\sum_{k=0}^{m}\left(\prod_{s=1}^{k}\frac{1}{r_s}\right)\left(\prod_{s=k+1}^{m} r_s\right)}, \quad i = 1, 2, \ldots, j \tag{49}$$

and

$$w_{ij} = -\frac{\left(\prod_{s=i}^{m} r_s\right)\sum_{k=1}^{j}\left(\prod_{s=1}^{k-1} r_s\right)\left(\prod_{s=k}^{j-1}\frac{1}{r_s}\right)}{a_{2j-1}\sum_{k=0}^{m}\left(\prod_{s=1}^{k}\frac{1}{r_s}\right)\left(\prod_{s=k+1}^{m} r_s\right)}, \quad i = j+1, j+2, \ldots, m+1, \tag{50}$$

where the quantities r_s are defined in (23).

Below is an example to illustrate Theorem 2.

Example. Consider $m \times m+1$ bidiagonal matrix

$$B = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}.$$

Calculations by formulas (49) and (50) give the following result:

$$B^+ = \frac{1}{m+1} \begin{bmatrix} m & m-1 & m-2 & \dots & 2 & 1 \\ -1 & m-1 & m-2 & \dots & 2 & 1 \\ -1 & -2 & m-2 & \dots & 2 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ -1 & -2 & -3 & \dots & 2 & 1 \\ -1 & -2 & -3 & \dots & -m+1 & 1 \\ -1 & -2 & -3 & \dots & -m+1 & -m \end{bmatrix}.$$

An Algorithm to Compute B^+. In Theorem 1 we give formulas for the elements of the matrix B^+. In addition, based on the expressions and recurrence relations obtained above, below we suggest a numerical procedure to compute the elements of the matrix $B^+ = [w_{ij}]_{m+1\times m}$.

Procedure MPInverse B^+.

1. Input elements $a_1, a_2, \dots, a_{2m}$ of the matrix B (see (2)).
2. Calculate the quantities r_s (see (23)):

$$r_s = a_{2s}/a_{2s-1}, \quad s = 1,2,\dots,m.$$

3. Calculate the quantities $\overset{\circ}{\mu}_i$ (see (21) and (25)):

$$\overset{\circ}{\mu}_{m+1} = 1, \quad \overset{\circ}{\mu}_i = r_i \overset{\circ}{\mu}_{i+1}, \quad i = m, m-1, \dots, 1.$$

4. Calculate the quantities $\overset{\circ}{\nu}_i$ (see (32) and (35)):

$$\overset{\circ}{\nu}_1 = 1, \quad \overset{\circ}{\nu}_{i+1} = \overset{\circ}{\nu}_i / r_i, \quad i = 1,2,\dots,m.$$

5. Calculate the quantities R_i (see (27)):

$$R_m = \frac{r_m}{a_{2m}^2}, \quad R_i = \frac{r_i}{a_{2i}^2}(a_{2i+1}^2 R_{i+1} + \overset{\circ}{\mu}_{i+1}), \quad i = m-1, m-2, \dots, 1.$$

6. Calculate the quantities S_i (see (37)):

$$S_1 = \frac{1}{a_1^2}, \quad S_i = \frac{1}{a_{2i-1}^2} \cdot \frac{1}{r_{i-1}}(a_{2i-2}^2 S_{i-1} + \overset{\circ}{\nu}_{i-1}), \quad i = 2,3,\dots,m.$$

7. Calculate the quantity

$$\sigma \equiv a_1^2 R_1 + \overset{\circ}{\mu}_1 .$$

8. Calculate the elements of the matrix B^+ (see (45) and (47)):

$$w_{ij} = \begin{cases} a_{2j-1} \overset{\circ}{\nu}_i R_j/\sigma, & i = 1,2,\dots,j, \\ -a_{2j-1} \overset{\circ}{\mu}_i S_j/\sigma, & i = j+1, j+2, \dots, m+1, \end{cases} \qquad j = 1,2,\dots,m.$$

9. Output the matrix $B^+ = [w_{ij}]_{m+1\times m}$.

End Procedure.

Direct calculations show that the numerical implementation of the procedure **MPInverse** B^+ requires $m^2 + O(m)$ arithmetical operations. By this fact, the algorithm may be considered as an optimal one.

The Elements of the Matrix A^+ (Odd Order). In this section we will show that the elements of the matrix B^+ obtained in Theorem 1 are, in fact, nonzero elements of the matrix A^+ for the odd order skew-symmetric matrix A from (1).

Let us return to equalities (5) and (6). We introduce the notation

$$Q^T B^+ P^T \equiv V = [v_{\alpha\beta}]_{2m+1\times 2m+1}.$$

Then, according to formula (6),

$$A^+ = V^T - V. \tag{51}$$

We also introduce an intermediate notation

$$Q^T B^+ \equiv U = [u_{\alpha j}]_{2m+1\times m}.$$

Then $V = UP^T$. Using definitions (3) and (4) of matrices P and Q, respectively, consider the following four options for the arrangement of elements in rows and columns of the matrix V.

• *Elements at the intersection of odd rows and odd columns of the matrix V.*

For $i = 1,2,\ldots,m+1$ and $j = 1,2,\ldots,m+1$:

$$v_{2i-1\,2j-1} = \sum_{k=1}^{m} u_{2i-1\,k}\, p_{2j-1\,k} = 0. \tag{52}$$

• *Elements at the intersection of odd rows and even columns of the matrix V.*

For $i = 1,2,\ldots,m+1$ and $j = 1,2,\ldots,m$:

$$v_{2i-1\,2j} = \sum_{k=1}^{m} u_{2i-1\,k}\, p_{2jk} = u_{2i-1\,j} = \sum_{k=1}^{m+1} q_{k\,2i-1} w_{kj} = w_{ij}. \tag{53}$$

• *Elements at the intersection of even rows and odd columns of the matrix V.*

For $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,m+1$:

$$v_{2i\,2j-1} = \sum_{k=1}^{m} u_{2ik}\, p_{2j-1\,k} = 0. \tag{54}$$

• *Elements at the intersection of even rows and even columns of the matrix V.*

For $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,m$:

$$v_{2i\,2j} = \sum_{k=1}^{m} u_{2ik}\, p_{2jk} = u_{2ij} = \sum_{k=1}^{m+1} q_{k\,2i} w_{kj} = 0. \tag{55}$$

Having expressions (52)–(55) and proceeding from representation (51) of the matrix A^+, we arrive at the following statement.

Theorem 2. *Let A be a skew-symmetric matrix of order $2m+1$ given in (1). It is assumed that $a_i \neq 0$, $1 \le i \le 2m$. The elements of the Moore–Penrose inverse $A^+ = [z_{\alpha\beta}]_{2m+1\times 2m+1}$ are as follows:*

$$z_{2i-1\,2j} = -w_{ij},\ z_{2j\,2i-1} = w_{ij}, \quad i = 1,2,\ldots,m+1,\ j = 1,2,\ldots,m, \tag{56}$$

where w_{ij} are the elements of the matrix B^+ obtained in Theorem 1; the remaining elements of the matrix A^+ are equal to zero.

As a presentation of the structure of matrix A^+, below we give the layout of its elements for the value $m = 3$:

$$A^+ = \begin{bmatrix} 0 & -w_{11} & 0 & -w_{12} & 0 & -w_{13} & 0 \\ w_{11} & 0 & w_{21} & 0 & w_{31} & 0 & w_{41} \\ 0 & -w_{21} & 0 & -w_{22} & 0 & -w_{23} & 0 \\ w_{12} & 0 & w_{22} & 0 & w_{32} & 0 & w_{42} \\ 0 & -w_{31} & 0 & -w_{32} & 0 & -w_{33} & 0 \\ w_{13} & 0 & w_{23} & 0 & w_{33} & 0 & w_{43} \\ 0 & -w_{41} & 0 & -w_{42} & 0 & -w_{43} & 0 \end{bmatrix}.$$

Conclusion. Summing up the results obtained in the article, we want to emphasize two main points. First, we have obtained closed form expressions for the elements of the Moore–Penrose inverse of odd order real tridiagonal skew-symmetric matrices. Secondly, on the basis of the obtained formulas and relations, a numerical algorithm which is optimal in terms of computational costs was constructed.

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ԵՌԵՔԱՆԿՅՈՒՆԱԳԾԱՅԻՆ ՇԵՂԱՍԻՄԵՏՐԻԿ ՄԱՏՐԻՑՆԵՐԻ ՄՈՒՐ-ՊԵՆՐՈՈՒԶԻ ՀԱԿԱԴԱՐՁՈՒՄԸ. II

Ներկա հոդվածը սույն ամսագրի նախորդ համարում հրապարակված [1] հոդվածի շարունակությունն է: Այսպեղ ներկայացված արդյունքները վերաբերում են Մուր–Պենրոուզի հակադարձ մապրիցի պարրերի բացահայտ արպահայպությունների սպացմանը՝ կենտ կարգի երեքանկյունագծային իրական շեղասիմեպրիկ

մատրիցների համար: Ստացված բանաձևերի հիման վրա կառուցված է ալգորիթմ, որի թվային իրականացումը պահանջում է օպտիմալ թվով թվաբանական գործողություններ:

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ОБРАЩЕНИЕ МУРА–ПЕНРОУЗА ТРЕХДИАГОНАЛЬНЫХ КОСОСИММЕТРИЧНЫХ МАТРИЦ. II

Настоящая статья является продолжением статьи [1], опубликованной в предыдущем номере журнала. Представленные здесь результаты касаются вывода явных выражений для элементов обратной матрицы Мура–Пенроуза в случае трехдиагональных вещественных кососимметричных матриц нечетного порядка. На основе полученных формул построен оптимальный, в смысле объема вычислительных затрат, численный алгоритм.