PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2023, **57**(2), p. 44–50

Mathematics

ON CORRECT SOLVABILITY OF DIRICHLET PROBLEM IN A HALF-SPACE FOR REGULAR EQUATIONS WITH NON-HOMOGENEOUS BOUNDARY CONDITIONS

M. A. KHACHATURYAN *

Chair of Mathematics and Mathematical Modeling, RAU, Armenia

In this paper we consider the following Dirichlet problem with non-homogeneous boundary conditions in a multianisotropic Sobolev space $W_2^{\mathfrak{M}}(R^2 \times R_+)$

$$\begin{cases} P(D_x, D_{x_3})u = f(x, x_3), \quad x_3 > 0, \quad x \in R^2, \\ D_{x_3}^s u\big|_{x_3=0} = \varphi_s(x), \quad s = 0, \dots, m-1. \end{cases}$$

It is assumed that $P(D_x, D_{x_3})$ is a multianisotopic regular operator of a special form with a characteristic polyhedron $\mathfrak{M}$. We prove unique solvability of the problem in the space $W_2^{\mathfrak{M}}(R^2 \times R_+)$, assuming additionally, that $f(x, x_3)$ belongs to $L_2(R^2 \times R^+)$ and has a compact support, boundary functions φ_s belong to special Sobolev spaces of fractional order and have compact supports.

https://doi.org/10.46991/PYSU:A/2023.57.2.044

MSC2020: 32Q40.

Keywords: regular operator, characteristic polyhedron, multianisotropic Sobolev space.

Introduction. In paper [1] a similar problem is considered with homogeneous boundary conditions in the multianisotropic Sobolev space $W_p^{\mathfrak{M}}(R^{n-1} \times R_+)$, where for a given completely regular polyhedron $\mathfrak{N}$, with principal vertices $\alpha^k \in Z_n^+$, $k = 0, 1, \dots, M$, the space $W_2^{\mathfrak{N}}(R^{n-1} \times R_+)$ is defined as follows [2]:

$$W_2^{\mathfrak{N}}(R^{n-1} \times R_+) =$$
$$\{f : f \in L_2(R^{n-1} \times R_+) \,\&\, D^{\alpha^k} f \in L_2(R^{n-1} \times R_+) \ \forall k = 0, \dots, M\},$$

with a norm

$$\|f\|_{W_2^{\mathfrak{N}}(R^{n-1} \times R_+)} = \|f\|_{L_2(R^{n-1} \times R_+)} + \sum_{k=0}^{M} \|D^{\alpha^k} f\|_{L_2(R^{n-1} \times R_+)}.$$

* E-mail: khmikayel@gmail.com

Using a special integral representation, containing all generalized derivatives of a function, corresponding to the vertices of the polyhedron $\mathfrak{M}$ (see [3–9]), in [1] an approximate solution for the problem with homogeneous boundary conditions was constructed and conditions for its unique solvability were obtained. In this paper, using the results from [10] related to the traces of functions from the multianisotropic Sobolev spaces, we obtain conditions, under which the problem with inhomogeneous boundary conditions is uniquely solvable in the space $W_2^{\mathfrak{M}}(R^2 \times R_+)$.

Basic Definitions and Notations. We denote by R_+ and Z_+ the set of non-negative real and integer numbers respectively. R^n is the n-dimensional real Euclidean space of points $x = (x_1, x_2, \dots, x_n)$ ($\xi = (\xi_1, \dots, \xi_n)$), Z_+^n is the set of n-dimensional multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \in Z_+$, $j = 1, 2, \dots, n$. For $x, \xi \in R^n$, $t \in Z_+$ and $\alpha \in Z_+^n$ denote $|\alpha| := \alpha_1 + \dots + \alpha_n$, $x\xi = \sum_{k=1}^{n} x_k \xi_k$, $\xi^t := (\xi_1^t, \xi_2^t, \dots, \xi_n^t)$, $\xi^\alpha := \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$, $D^\alpha := D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} \cdots D_{x_n}^{\alpha_n}$, where $D_{x_j} = i^{-1} \dfrac{\partial}{\partial x_j}$ ($i^2 = -1$) is the generalized differentiation operator according to S.L. Sobolev.

For a given set of multi-indexes $A \subset Z_+^n$ denote by $\mathfrak{N} = \mathfrak{N}(A)$ the smallest convex polyhedron, containing all points in A. Polyhedron $\mathfrak{N}$ is called completely regular, if it has a vertex at the origin, a vertex on each coordinate axis, different from the origin, and the outer normals of all $(n-1)$-dimensional non-coordinate faces have positive coordinates. A vertex of a completely regular polyhedron, different from the origin is called a principal vertex. The set of all principal vertices is denoted by $\partial'\mathfrak{N}$.

For a given differential operator $P(D) = \sum \gamma_\alpha D^\alpha$, denote $(P) := \{\alpha \in Z_+^n, \gamma_\alpha \neq 0\}$. Polyhedron $\mathfrak{N} = \mathfrak{N}(\{0\} \bigcup (P))$ is called the characteristic polyhedron of the operator $P(D)$. An operator $P(D)$ is said to be regular, if for some constant $C > 0$

$$|P(\xi)| \geq C \sum_{\alpha \in \partial'\mathfrak{N}} |\xi^\alpha|, \ \ \forall \xi \in R^n.$$

In paper [10] two dimensional Sobolev spaces of fractional order are considered. Let $\mathfrak{N} \subset R^2$ be a completely regular polyhedron, $q > 0$ be an arbitrary rational number, $K_{q,\mathfrak{N}}(\xi) := 1 + \sum_{k=0}^{M} (\xi^2)^{q\alpha^k}$. Through $W_2^{q\mathfrak{N}}(R^2)$ we denote the Sobolev space of fractional order, defined by

$$W_2^{q\mathfrak{N}}(R^2) := \{u : u \in L_2(R^2) \,\&\, \sqrt{K_{q,\mathfrak{N}}(\xi)} F[u](\xi) \in L_2(R^2)\}$$

with a norm

$$\|u\|_{W_2^{q\mathfrak{N}}(R^2)} = \left(\int_{R^2} K_{\mathfrak{N}}(\xi) |F[u](\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

where $F[u]$ is the Fourier transform of function u.

Let $\mathfrak{N} \subset R^2$ be a two-dimensional completely regular polyhedron with principal vertices $\alpha^0 := (l_1, 0), \alpha^1, \dots, \alpha^M := (0, l_2)$, enumerated counterclockwise. Denote by μ^i the outward normal to the side of the polyhedron, passing through the vertices α^{i-1}, α^i $(i = 1, \dots, M)$, normalized in such a way, that the line, passing through this side of the polyhedron, is described by the equation $(\mu^i, t) = 1$, $t \in R^2$.

Statement of the Main Results. Consider the following boundary value problem in a half-space:

$$\begin{cases} P(D_x, D_{x_3})u = f(x, x_3), \quad x_3 > 0, \quad x \in R^2, \\ D_{x_3}^s u\big|_{x_3=0} = \varphi_s(x), \quad s = 0, \dots, m-1. \end{cases} \tag{1}$$

Let us define the conditions imposed on the operator $P(D_x, D_{x_3})$.

1) Differential operator $P(D_x, D_{x_3})$ has the form

$$P(D_x, D_{x_3}) = D_{x_3}^{2m} + \sum_{i=0}^{M} a_i D_x^{\alpha^i} := D_{x_3}^{2m} + P_0(D_x)$$

with constant real coefficients $a_i \neq 0$ $(i = 0, \dots, M)$, $m \in N$, $\alpha^i \in Z_+^2$ $(i = 0, \dots, M)$.

2) The characteristic polyhedron $\mathfrak{M}$ of the operator $P(D_x, D_{x_3})$ is completely regular.

3) The operator $P(D_x, D_{x_3})$ is regular.

Denote

$$\mu^0 := \left(\frac{1}{l_1}, \frac{1}{l_2}\right), \quad \chi := \frac{1}{2}\left(|\mu^0| + \frac{1}{2m}\right),$$

$$q(s) := 1 - \frac{s}{2m} - \frac{1}{4m}, \quad s = 0, \dots, m-1.$$

Theorem 1. *Let the operator $P(D_x, D_{x_3})$ satisfy conditions 1)–3). If $f \in L_2(R^2 \times R_+)$ has a compact support, $\varphi_s \in W_2^{q(s)\mathfrak{N}}(R^2)$ and has a compact support ($s = 0, \dots, m-1$), then for $\chi > 1$ problem (1) has a unique solution U from the class $W_2^{\mathfrak{M}}(R^2 \times R_+)$, and with some constant $C > 0$ (depending only on supp(f), supp(φ_s)) it holds the inequality*

$$\|U\|_{W_2^{\mathfrak{M}}(R^2 \times R_+)} \leq C \left(\|f\|_{L_2(R^2 \times R_+)} + \sum_{s=0}^{m-1} \|\varphi_s\|_{W_2^{q(s)\mathfrak{N}}(R^2)} \right). \tag{2}$$

When $\chi \leq 1$ the following theorem holds.

Theorem 2. *Let $\chi \leq 1$ and the conditions of Theorem 1 hold. If the function f satisfies the orthogonality conditions*

$$\int_{R^2} x^{\alpha} f(x, x_3) dx = 0$$

for $|s| = 0, 1, \dots, L-1$, where L is a natural number determined from the inequality

$$\chi + L \min_{i=1,2} \mu_i^0 > 1 \geq \chi + (L-1) \min_{i=1,2} \mu_i^0,$$

then problem (1) has a unique solution from the class $W_2^{\mathfrak{M}}(R^2 \times R_+)$ for which inequality (2) holds.

Remark. *Conditions, put on variable χ in Theorems 1, 2, as well as the orthogonality condition on function f in Theorem 2 is not explicitly used in this paper, rather they are used in [1] in order to prove the unique solvability of problem with homogeneous boundary conditions and to obtain the estimate of Sobolev norm of solution U by L_2 norm of f.*

Proof of the Main Results. Let the above notations hold.

Lemma. *For any given collection of functions $\varphi_s \in W_2^{q(s)\mathfrak{N}}(R^2)$, $s = 0,1,\ldots,m-1$, having a compact support, there exists a function $F \in W_2^{\mathfrak{M}}(R^3)$ with a compact support, which satisfies the following properties.*

$$D_{x_3}^s F\big|_{x_3=0} = \varphi_s,\ \forall s = 0,1,\ldots,m-1, \tag{3}$$

$$\|F\|_{W_2^{\mathfrak{M}}(R^3)} \le C \sum_{s=0}^{m-1} \|\varphi_s\|_{W_2^{q(s)\mathfrak{N}_0}(R^2)}, \tag{4}$$

where $C > 0$ is a constant, depending only on supp(φ_s).

Proof. It follows from Theorem 3.3 in [10], that there exists a function $F_0 \in W_2^{\mathfrak{M}}(R^3)$(not necessarily with a compact support), which satisfies (3) and (4) with some constant $C_0 > 0$, independent from φ_s. Let Ω be any open, bounded set which contains $\bigcup_{s=0}^{m-1}$ supp(φ_s), and let $g \in C_0^\infty(R^3)$ be a function with compact support, such that $g \equiv 1$ on $\Omega \times (-1,1)$. Let's prove that $F := F_0 \cdot g$, which also belongs to $W_2^{\mathfrak{M}}(R^3)$ and has a compact support, satisfies (3) and (4). Indeed,

$$D_{x_3}^s F\big|_{x_3=0} =$$

$$D_{x_3}^s (F_0 \cdot g)\big|_{x_3=0} = \sum_{i=0}^{s} \varphi_i \cdot D_{x_3}^{s-i} g\big|_{x_3=0} = \varphi_s.$$

As for the estimate of the norm, we have

$$\|F_0 \cdot g\|_{W_2^{\mathfrak{M}}(R^3)} \le C_1 \|F_0\|_{W_2^{\mathfrak{M}}(R^3)} \le C_1 \cdot C_0 \sum_{s=0}^{m-1} \|\varphi_s\|_{W_2^{q(s)\mathfrak{N}_0}(R^2)},$$

so F satisfies (4) with constant $C = C_0 \times C_1$, depending only on supp(φ_s).

Lemma 1 is proved. □

Proof of Theorems 1, 2. Denote $\overline{f} := f - P(D_x, D_{x_3})F$, where $F \in W_2^{\mathfrak{M}}(R^3)$ and $D_{x_3}^s F\big|_{x_3=0} = \varphi_s$ (see Lemma 1). Consider the following problem with the homogeneous boundary conditions

$$\begin{cases} P(D_x, D_{x_3})u = \overline{f}(x, x_3), \quad x_3 > 0, \quad x \in R^2, \\ D_{x_3}^s u\big|_{x_3=0} = 0, \quad s = 0,\ldots,m-1. \end{cases} \tag{5}$$

According to Theorems 1.1 and 1.2 of [1] problem (5) has a solution $\overline{U} \in W_2^{\mathfrak{M}}(R^2 \times R_+)$, for which the following relations hold:

$$D_{x_3}^s \overline{U}\big|_{x_3=0} = 0,\ s = 0,1,\ldots,m-1,$$

$$\|\overline{U}\|_{W_2^{\mathfrak{M}}(R^2 \times R_+)} \le C_0 \|\overline{f}\|_{L_2(R_3^+)},$$

where $C_0 > 0$ is a constant depending on $\mathrm{supp}(\overline{f})$. Let us prove that the function $U := \overline{U} + F$ is a solution to problem (1) satisfying (2). Indeed.

$$P(D_x, D_{x_3})U = P(D_x, D_{x_3})(\overline{U} + F) =$$
$$f - P(D_x, D_{x_3})F + P(D_x, D_{x_3})F = f,$$
$$D^s_{x_3}U\big|_{x_3=0} = D^s_{x_3}\overline{U}\big|_{x_3=0} + D^s_{x_3}F\big|_{x_3=0} = \varphi_s,\ s = 0, 1, \dots, m-1.$$

Let us show that U satisfies the inequality (2).

$$\|U\|_{W_2^{\mathfrak{M}}(R^2\times R_+)} \leq \|\overline{U}\|_{W_2^{\mathfrak{M}}(R^2\times R_+)} + \|F\|_{W_2^{\mathfrak{M}}(R^2\times R_+)} \leq$$
$$C_0\|f - P(D_x, D_{x_3})F\|_{L_2(R^2\times R_+)} + \|F\|_{W_2^{\mathfrak{M}}(R^3)}.$$

Since with some constant $C_1 > 0$ the inequality

$$\|P(D_x, D_{x_3})F\|_{L_2(R^3)} \leq C_1 \cdot \|F\|_{W_2^{\mathfrak{M}}(R^3)}$$

holds, taking into account Lemma 1, we have

$$\|U\|_{W_2^{\mathfrak{M}}(R^3_+)} \leq C_2 \left(\|f\|_{L_2(R^2\times R_+)} + \sum_{s=0}^{m-1} \|\varphi_s\|_{W_2^{q(s)\mathfrak{M}}(R^3)} \right).$$

The uniqueness of the solution is proved in the same way as in Theorems 1.1, 1.2 in [1].

Theorems 1, 2 are proved. □

Author expresses his deepest gratitude to the journal reviewer for the meticulous work he has dedicated to reviewing and providing feedback on the article. His expertise, attention to detail, and insightful comments have greatly contributed to improving the quality and clarity of research.

Received 16.05.2023
Reviewed 01.06.2023
Accepted 16.06.2023

REFERENCES

1. Karapetyan G.A., Petrosyan H.A. Correct Solvability of the Dirichlet Problem in the Half-space for Regular Hypoelliptic Equations. *Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)* **54** (2019), 45–69. https://doi.org/10.3103/S1068362319040022
2. Ghazaryan H.G. The Newton Polyhedron, Spaces of Differentiable Functions and General Theory of Differential Equations. *Armenian Journal of Mathematics* **9** (2017), 102–145.
3. Karapetyan G.A. Integral Representations of Functions and Embedding Theorems for Multianisotropic Spaces on the Plane with One Anisotropy Vertex. *Journal of Contemporary Mathematical Analysis* **51** (2016), 269–281. https://doi.org/10.3103/S1068362316060017

4. Karapetyan G.A. Integral Representation of Functions and Embedding Theorems for Multianisotropic Spaces for the Three-dimensional Case. *Eurasian Mathematical Journal* **7** (2016), 19–39.
5. Karapetyan G.A., Arakelyan M.K. Estimation of Multianisotropic Kernels and their Application to the Embedding Theorems. *Transactions of A. Razmadze Mathematical Institute* **171** (2017), 48–56.
6. Karapetyan G.A. Integral Representations of Functions and Embedding Theorems for *n*-Dimensional Multianisotropic Spaces with One Anisotropy Vertex. *Siberian Mathematical Journal* **58** (2017), 445–460.
https://doi.org/10.1134/S0037446617030089
7. Karapetyan G.A., Petrosyan H.A. Embedding Theorems for Multianisotropic Spaces with Two Vertices of Anisotropicity. *Proc. of the YSU. Physical and Mathematical Sci.* **51** (2017), 29–37.
https://doi.org/10.46991/PYSU:A/2017.51.1.029
8. Karapetyan G.A. An Integral Representation and Embedding Theorems in the Plane for Multianisotropic Spaces. *Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)* **52** (2017), 267–275.
https://doi.org/10.3103/S1068362317060024
9. Karapetyan G.A., Arakelyan M.K. Embedding Theorems for General Multianisotropic Spaces. *Matematical Notes* **104**(2018), 422–438.
https://doi.org/10.1134/S0001434618090092
10. Khachaturyan M.A., Hakobyan A.R. On Traces of Functions from Multianisotropic Sobolev Spaces. *Vestnik RAU. Phys.-Math. Est. Nauki* **1** (2021), 56–77 (in Russian).

Մ. Ա. ԽԱՉԱՏՈՒՐՅԱՆ

ԱՆՀԱՄԱՍԵՌ ԵԶՐԱՅԻՆ ՊԱՅՄԱՆՆԵՐՈՎ ԴԻՐԻԽԼԵՅԻ ԽՆԴՐԻ ՆՈՐՄԱԼ ԼՈՒԾԵԼԻՈՒԹՅՈՒՆԸ ԿԻՍԱՏԱՐԱԾՈՒԹՅՈՒՆՈՒՄ ՌԵԳՈՒԼՅԱՐ ՀԱՎԱՍԱՐՄԱՆ ՀԱՄԱՐ

Աշխատանքում ուսումնասիրվում է հետևյալ Դիրիխլեի խնդիրը՝ անհամասեռ եզրային պայմաններով, $W_2^{\mathfrak{M}}(R^2 \times R_+)$ Սոբոլևյան տարածությունում

$$\begin{cases} P(D_x, D_{x_3})u = f(x, x_3), \quad x_3 > 0, \quad x \in R^2, \\ D_{x_3}^s u\big|_{x_3=0} = \varphi_s(x), \quad s = 0, \dots, m-1 : \end{cases}$$

Ենթադրվում է, որ $P(D_x, D_{x_3})$-ը հատուկ տեսքի ռեգուլյար մուլտիանիզոտրոպ օպերատոր է $\mathfrak{M}$ բնութագրիչ բազմանիստով:

Հավելյալ ենթադրելով, որ $f(x, x_3)$-ը $L_2(R^2 \times R^+)$-ից կոմպակտ կրիչով ֆունկցիա է, φ_s եզրային ֆունկցիաները պատկանում են հատուկ կոտորակային կարգի Սոբոլևյան տարածությունների և ունեն կոմպակտ կրիչներ, ապացուցված է խնդրի եզակի լուծելիությունը $W_2^{\mathfrak{M}}(R^2 \times R_+)$ դասում:

М. А. ХАЧАТУРЯН

НОРМАЛЬНАЯ РАЗРЕШИМОСТЬ ЗАДАЧИ ДИРИХЛЕ С НЕОДНОРОДНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ В ПОЛУПРОСТРАНСТВЕ ДЛЯ РЕГУЛЯРНЫХ УРАВНЕНИЙ

В работе рассматривается следующая задача Дирихле с неоднородными граничными условиями в мультианизотропном пространстве Соболева $W_2^{\mathfrak{M}}(R^2 \times R_+)$:

$$\begin{cases} P(D_x, D_{x_3})u = f(x, x_3), \quad x_3 > 0, \quad x \in R^2, \\ D_{x_3}^s u\big|_{x_3=0} = \varphi_s(x), \quad s = 0, \dots, m-1. \end{cases}$$

Предполагается, что $P(D_x, D_{x_3})$– мультианизоторпный регулярный оператор специального вида с характеристическим многогранником $\mathfrak{M}$.

Предполагая дополнительно, что $f(x, x_3)$ – функция из $L_2(R^2 \times R^+)$ с компактным носителем, граничные функции φ_s принадлежат специальным пространствам Соболева дробного порядка и имеют компактные носители, доказана однозначная разрешимость задачи в пространстве $W_2^{\mathfrak{M}}(R^2 \times R_+)$.