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COMMUNICATIONS

Mathematics

ON ONE FORMULA OF TRACES

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The paper generalizes one formula of traces, established by J. Neumann for square matrices, for nuclear operators.

***Keywords*:** singular number, trace formula.

Preliminaries. Let H be a separable Hilbert space. We will denote by $BL(H)$ the Banach algebra of all bounded linear operators acting in the space H, and let J_∞ be the ideal of all compact operators acting in H. For operator $A \in BL(H)$ we will denote by $|A|$ the unique nonnegative square root of the operator $A^* A$. It is obvious that the compactness of one of the operators $A,\ A^*A,\ |A|$ implies the compactness of remaining two operators. Let $A \in J_\infty$. As $|A|$ is compact, self-adjoint and nonnegative, its nonzero eigenvalues can be rearranged in decreasing order. We will denote by $s_j(A)$ the j-th eigenvalue of the operator $|A|$ (note that each eigenvalue is counted with multiplicity). Numbers $s_j(A)$ are called the singular numbers of the operator A (see [1, 2]). For $p \in [1, \infty)$ we will denote by J_p the set of all operators $A \in J_\infty$, satisfying to the condition $\sum_{j=1}^{\infty} s_j^p(A) < \infty$. Then the formula

$$\|A\|_p = \left[\sum_{j=1}^{\infty} s_j^p(A)\right]^{\frac{1}{p}} \tag{1}$$

defines a norm in J_p and with respect to this norm J_p is a separable Banach space. J_p is also two-sided ideal of $BL(H)$ and has the following property of symmetry: for $A \in J_p$ we have $A^* \in J_p$ and $\|A^*\|_p = \|A\|_p$ (see [1]). Clearly, all the propositions about $J_p\ (1 \le p < \infty)$ are true also for J_∞. Elements of J_1 are called nuclear operators.

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Let's note also the following property of ideals J_p : if $A \in J_p$ $(1 \le p \le \infty)$ and $B \in J_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, then $AB, BA \in J_1$.

Now we will give the definition and some properties of the trace of a nuclear operator. Let $A \in J_1$, and $\{e_n\}$ be an orthonormal (finite or countable) base of the space H. Then the series $\sum_n (Ae_n, e_n)$ converges, and its sum does not depend on a choice of the orthonormal base $\{e_n\}$. This sum is called (matrix) trace of operator A and is denoted by $\mathrm{tr}(A)$. If $A \in J_\infty$, $B \in BL(H)$ and $AB, BA \in J_1$, then

$$\mathrm{tr}(AB) = \mathrm{tr}(BA). \tag{2}$$

The norm of J_2 and the trace are connected by the following relation (see [3]):

$$\|A\|_2^2 = \mathrm{tr}\left(A^* A\right) \quad (A \in J_2). \tag{3}$$

The Formula of Traces. We will establish a relation, connecting norms of commutators $AB - BA$ and $AB^* - B^* A$.

Theorem. If one of the bounded linear operators A, B, acting in Hilbert space H, is nuclear, then the following equality holds:

$$\|AB - BA\|_2^2 - \left\|AB^* - B^* A\right\|_2^2 = -\mathrm{tr}\left[\left(A^* A - AA^*\right)\left(B^* B - BB^*\right)\right]. \tag{4}$$

Proof. First notice, that by the conditions of the Theorem operators $AB - BA$ and $AB^* - B^* A$ are nuclear, and since for $1 \le r \le s \le \infty$ the following inclusion $J_s \subset J_r$ is true, the left part of (4) is well-defined. We will denote

$$\rho = \|AB - BA\|_2^2 - \left\|AB^* - B^* A\right\|_2^2 + \mathrm{tr}\left[\left(A^* A - AA^*\right)\left(B^* B - BB^*\right)\right].$$

In view of (3), we have

$$\rho = \mathrm{tr}\left[(AB - BA)^* (AB - BA)\right] - \mathrm{tr}\left[\left(AB^* - B^* A\right)^* \left(AB^* - B^* A\right)\right] +$$
$$+\mathrm{tr}\left[\left(A^* A - AA^*\right)\left(B^* B - BB^*\right)\right] = \mathrm{tr}\left(B^* A^* AB\right) - \mathrm{tr}\left(B^* A^* BA\right) -$$
$$-\mathrm{tr}\left(A^* B^* AB\right) + \mathrm{tr}\left(A^* B^* BA\right) - \mathrm{tr}\left(BA^* AB^*\right) + \mathrm{tr}\left(BA^* B^* A\right) + \mathrm{tr}\left(A^* BAB^*\right) -$$
$$-\mathrm{tr}\left(A^* BB^* A\right) + \mathrm{tr}\left(A^* AB^* B\right) - \mathrm{tr}\left(A^* ABB^*\right) - \mathrm{tr}\left(AA^* B^* B\right) + \mathrm{tr}\left(AA^* BB^*\right)$$

or

$$\rho = \left[\mathrm{tr}\left(B^* A^* AB\right) - \mathrm{tr}\left(A^* ABB^*\right)\right] + \left[\mathrm{tr}\left(A^* B^* BA\right) - \mathrm{tr}\left(AA^* B^* B\right)\right] +$$
$$+\left[\mathrm{tr}\left(BA^* B^* A\right) - \mathrm{tr}\left(A^* B^* AB\right)\right] + \left[\mathrm{tr}\left(A^* BAB^*\right) - \mathrm{tr}\left(B^* A^* BA\right)\right] +$$
$$+\left[\mathrm{tr}\left(A^* AB^* B\right) - \mathrm{tr}\left(BA^* AB^*\right)\right] + \left[\mathrm{tr}\left(AA^* BB^*\right) - \mathrm{tr}\left(A^* BB^* A\right)\right].$$

According to (2), the expressions, standing in the square brackets, are equal to zero, and consequently $\rho = 0$, i.e. (4) is true.

The Theorem is proved.

Remark 1. The steps in the proof of the Theorem show, that the statement of the Theorem remains true, if instead of nuclearity of operators A or B we assume, that one of these operators belongs to J_p, and another to J_q, where $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2. The formula (4) generalizes the similar formula for the square matrices, established by J. Neumann (see [4, 5]). If in the conditions of the Theorem one of operators A, B is normal, then the commutability of A, B implies the commutability of A, B^*. This fact for square matrices has been noticed by Neumann, who has raised the question about its possible generalisations. In 1950 Fuglede, Putnam and Rosenblum have shown that this statement is true for normal operators from algebra $BL(H)$ (see [6]). This result of Fuglede, Putnam and Rosenblum, which goes back to Neumann, has far-reaching generalizations, which can be found in works [7, 8].

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Հ. Ա. Ասատրյան

Հետքերի մի բանաձևի մասին

Ջ. Նեյմանի կողմից քառակուսային մատրիցների համար ստացված հետքերի մի բանաձև ընդհանրացվում է միջուկային օպերատորների համար:

Г. А. Асатрян.

Об одной формуле следов

Одна формула следов, установленная Дж. Нейманом для квадратных матриц, обобщается для ядерных операторов.